$$(A + B)C \leqq AC + BC,$$

and our theorem is proved.

We can also prove the following:

3.2. COROLLARY. *A necessary and sufficient condition that*

$$(A + B)C = AC + BC$$

for positive A, B, *and* C *is that either* $C=1$, *or* $1<C<\omega$ *and* $\alpha_0 \leqq \beta_0$, *or* $\omega \leqq C$ *and* $\alpha_0+\gamma_0<\beta_0+\gamma_0$.

This corollary follows quite easily from the reasoning found in the preceding section.

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THE DECOMPOSITION THEOREM FOR ABELIAN GROUPS[1]

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Let G be an abelian group such that $p^k g=0$ for all $g \in G$, p prime, k fixed. We prove G *has a basis*, that is, a set of elements such that each $g \in G$ is uniquely expressible as a linear combination of elements of the set.[2]

THEOREM. *There exists an ascending chain of sets* B_i, $0 \leqq i \leqq k$, *of elements of* G *with the properties*:

(i) *Every element in* B_i *is of order greater than* p^{k-i}.

(ii) *The elements in* B_i *are completely linearly independent.*

(iii) *If the order of the element* g *in* G *is greater than* p^{k-i}, *then there exists a (unique) linear combination* z *of elements of* B_i *such that the order of* $g-z$ *is at most* p^{k-i}.

Since we may choose as B_0 the vacuous set, we may assume that the sets $B_0, \cdots, B_s$ have already been constructed in such a way as to meet the requirements (i) to (iii). In order to construct B_{s+1} we adjoin to B_s any greatest subset C of G with the following properties.

(a) All the elements in C are of order p^{k-s}.

(b) The join B_{s+1} of the sets B_s and C is an independent set.

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[1] Presented to the Society, April 6, 1940.

[2] Unique in that the number of nonzero terms in an expression for g is unique and only the arrangement but not the respective values of the nonzero terms may differ in two expressions for g.

The set B_{s+1} satisfies (i) and (ii). In order to prove (iii) let h be any element in G whose order is at least p^{k-s}. If firstly the order of h is exactly p^{k-s}, then it follows from the conditions (a) and (b) that there exists an integer m and a linear combination $y=\sum a_j y_j$ of elements in B_{s+1} so that $0\neq mh=y$. Assume without loss of generality that m is a power of $p<p^{k-s}$. Then $(p^{k-s}/m)mh=0=\sum a_j(p^{k-s}/m)y_j$. By (ii), p^{k-s} divides $a_j(p^{k-s}/m)$, so that a_j/m is an integer (all j). Therefore the order of $h-\sum(a_j/m)y_j$ is $m<p^{k-s}$. If secondly the order of h is greater than p^{k-s}, then there exists a linear combination z of elements in B_s such that the order of $h-z$ is at most p^{k-s}, and thus (as above) $y=\sum c_j y_j$ can be found with y_j in B_{s+1} for which $h-z-y$ has order $<p^{k-s}$. We have shown[3] that B_k is a basis of G.

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[3] The following contain proofs for finite groups: Andreas Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3d edition, Berlin, Julius Springer, 1937, p. 46–49. R. Remak, *Über die Zerlegung der kommutativen Gruppen in zyklische teilerfremde Faktoren*, Journal für die reine und angewandte Mathematik, vol. 141. L. C. Mathewson, *A simple proof of a theorem of Kronecher*, ibid., vol. 161 (1929), p. 255. A. Korselt, ibid. N. Tschebotaroew, *Bewies der Existenz einer Basis bei Abelschen Gruppen von endlicher Ordnung*, Kasan University, Fizzico-Matematico Obchestvo Izviestiia, vol. 4 (1929–1930). Third line after δ) in the proof read $\omega=\omega_s r$ for $\omega=\omega_s=r$. In the fourth paragraph following, read $\bar{\omega}_s=q\omega_s+t$, $0\leqq t<\omega$, $A_s\cdots t\cdots$. The following contain proofs for infinite groups: H. Pruefer, *Untersuchungen ueber die Zerlegbarkeit*, Mathematische Zeitschrift, vol. 17 (1923), p. 53; R. Baer, Compositio Mathematica, vol. 1 (1934), p. 274.