ON THE EQUATION $dy/dx=f(x, y)$[1]

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We consider here the differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is a one-valued function defined on an open region R of the xy-plane. By a solution curve of this equation we mean a curve $y=y(x)$ which has a derivative at every point and which satisfies everywhere the differential equation. There are known sufficiency conditions on f for the existence of a one parameter family of solution curves simply covering R. But, as Professor Bamforth mentioned to me orally, there seems to be in the literature no corresponding necessary conditions. We shall prove that one necessary condition is that f be the limit of a sequence of continuous functions.[2]

A curve is the xy-plane will be termed a *continuous function curve* if the points of the curve are the points $(x, \phi(x))$, $a<x<b$, where $\phi(x)$ is a one-valued continuous function defined on an open interval (a, b). An open region R of the xy-plane will be said to be simply covered by a set $\mathcal{J}$ of such curves if:

(1) Every point of R is on one and only one curve of $\mathcal{J}$.

(2) Every curve of $\mathcal{J}$ stretches from boundary to boundary of R; that is, if S is any set of points on a curve C of $\mathcal{J}$, each limit point of S is either itself a point of C or a boundary point of R.

THEOREM. *If an open region R of the xy-plane is simply covered by a set $\mathcal{J}$ of continuous function curves $y=\phi(x)$, then for every point (x_0, y_0) of R there exists an open subregion R_0 of R containing (x_0, y_0) such that the family of curves constituted by the portions of the curves of $\mathcal{J}$ in R_0 is representable by the equation $y=\phi(x, a)$, where ϕ is a continuous function of x and the parameter a.*

PROOF. Let (x_0, y_0) be a point of such a given region R; R_1 a rectangle interior to R with (x_0, y_0) as center; and h a positive number such that the points (x_0, y_0-h) and (x_0, y_0+h) are inside R_1. Since $\mathcal{J}$ simply covers R, there exist curves $y=\phi_1(x)$ and $y=\phi_2(x)$ of $\mathcal{J}$ containing the points (x_0, y_0-h) and (x_0, y_0+h) respectively. Also, the continuity of $\phi_1(x)$ and $\phi_2(x)$ insures the existence of an open interval I containing x_0 such that the points of the curves $y=\phi_1(x)$ and

[1] I wish to express my thanks to Professor Henry Blumberg for his aid in the preparation of this paper.

[2] Of course, as is well known, the derivative $f'(x)$ of a function $f(x)$ has this property. It may be expected that $f(x, y)$ necessarily has also other properties corresponding to known properties of $f'(x)$.

$y=\phi_2(x)$, with x in I, lie in R_1. If (x_1, x_2) is a closed interval interior to I and containing x_0, we have $\phi_1(x)<\phi_2(x)$ for $x_1\leqq x\leqq x_2$, since $\phi_1(x_0)<\phi_2(x_0)$ and no two curves of $\mathcal{J}$ cross each other. We denote by R_0 the open region of points (x, y) bounded by the parallels $x=x_1$, $x=x_2$, and the curves $y=\phi_1(x)$, $y=\phi_2(x)$. In view of conditions (1) and (2) it readily follows that the interval of definition of any curve $y=\phi(x)$ of $\mathcal{J}$ having points in R_0 contains the entire closed interval (x_1, x_2).

We now define the function $\phi(x, a)$, mentioned in the theorem, by letting $y=\phi(x, a)$ be the equation of the curve of $\mathcal{J}$ passing through the point (x_1, a) on the boundary of R_0. Thus $\phi(x, a)$ is defined in the region $\overline{R}_0$ of the xa-plane, $x_1<x<x_2$, $\phi_1(x_1)<a<\phi_2(x_1)$. The family of curves $y=\phi(x, a)$, where x and the parameter a take on values in $\overline{R}_0$, is identical with the set consisting of the portions of the curves of $\mathcal{J}$ which are interior to R_0. $\phi(x, a)$ is a continuous function of x and a in $\overline{R}_0$. For let (ξ, α) be a point of this region, ϵ any positive number, and α_1, α_2 numbers such that $y=\phi(x, \alpha_1)$, $y=\phi(x, \alpha_2)$ are respectively equations of the curves of $\mathcal{J}$ which pass through the points $(\xi, \phi(\xi, \alpha)-\epsilon')$, $(\xi, \phi(\xi, \alpha)+\epsilon')$ of R_0, ϵ' being a positive number less than $\epsilon/4$. Due to the continuity of $\phi(x, \alpha)$, considered as a function of x, we may find an interval I_1, containing ξ, such that $\phi(x, \alpha_1)$ differs from $\phi(\xi, \alpha_1)$ by an amount less than $\epsilon/4$ for values of x in I_1. Also, we may choose an interval I_2 such that $\phi(x, \alpha_2)$ differs from $\phi(\xi, \alpha_2)$ by less than $\epsilon/4$ for x in I_2. Thus $\phi(x, a)$ differs from $\phi(\xi, \alpha)$ by less than ϵ if $\alpha_1<a<\alpha_2$ and x is in I_1I_2—the interval consisting of the points common to both I_1 and I_2. Consequently, $\phi(x, a)$ is continuous in $\overline{R}_0$, and the theorem is proved.

LEMMA. *The function $\phi(x, a)$ has a continuous inverse function $a(x, y)$ defined on R_0, which satisfies the identity $\phi(x, a(x, y))\equiv y$.*

PROOF. We associate with every point (x, y) of R_0 that value $a=a(x, y)$ such that the curve $y=\phi(x, a)$ passes through the point (x, y). In short, $\phi(x, a(x, y))=y$. The function $a(x, y)$ thus defined is continuous in R_0. For suppose it were discontinuous at a point (ξ, η) of R_0. Since $\phi(x, a)$ is continuous at $(\xi, a(\xi, \eta))$, and properly monotone in a, the composite function $\phi(x, a(x, y))$, considered as a function of x and y, is discontinuous at (ξ, η). But $\phi(x, a(x, y))\equiv y$ for points (x, y) in R_0 and is therefore continuous at the point (ξ, η), contrary to assumption. Therefore $a(x, y)$ is continuous in R_0.

THEOREM. *For the existence of a family $\mathcal{J}$ of continuous function curves simply covering R which are solution curves of the equation*

$dy/dx=f(x, y)$, where R is an open region on which f is defined, it is necessary that f be the limit of a sequence of continuous functions.

PROOF. We consider $f(x, y)$ given, and assume that a family $\mathcal{J}$, as described, exists. If (x_0, y_0) is a point of R there may be determined, as we have shown, an open region R_0 containing the point such that the portions of the curves of $\mathcal{J}$ in R_0 are the curves $y=\phi(x, a)$, $x_1<x<x_2, a_1<a<a_2$. $\phi(x, a)$ has, according to the above lemma, a continuous inverse function $a(x, y)$ defined on R_0 for which $\phi(x, a(x, y))=y$. Let R_0' be the open subregion consisting of the points (x, y) of R_0 where $x_1<x<x_2-k$, k being a positive number less than x_2-x_0. If (ξ, η) is a point of R_0', the equation of the curve of $\mathcal{J}$ which passes through this point is $y=\phi(x, \alpha)$, where $\alpha=a(\xi, \eta)$. By hypothesis, $y=\phi(x, \alpha)$ is a solution curve of the differential equation $dy/dx=f(x, y)$. At (ξ, η) this equation may be written:

$$f(\xi, \eta) = \lim_{n\to\infty} \{\phi(\xi + k/n, a(\xi, \eta) - \phi(\xi, a(\xi, \eta))\}/(k/n)$$

where n is a positive integer. For convenience, we denote the difference quotient present in the right-hand member of this equation by $\psi_n(\xi, \eta)$. Inasmuch as (ξ, η) is a general element of R_0', we have $f(x, y)=\lim_{n\to\infty} \psi_n(x, y)$ in this region. Moreover, $\psi_n(x, y)$ is continuous since $a(x, y)$ is continuous in R_0' and $\phi(x, a)$ is continuous at points $(x, a(x, y))$ and $(x+k/n, a(x, y))$, where (x, y) is in R_0'. We conclude that for every point (x_0, y_0) of R, there exists an open region R_0' containing (x_0, y_0) such that in R_0' the function $f(x, y)$ is the limit of a sequence of continuous functions.

It follows that f is the limit of a sequence of continuous functions in its entire region of definition R. For let P be any perfect subset of the points of the xy-plane containing points of R, and (x_0, y_0) any point of PR. As we have seen, there exists an open region R_0' containing (x_0, y_0) such that, in this region, f is the limit of a sequence of continuous functions. The latter statement implies, it may be shown, that PR_0' has a point of continuity of f with respect to PR_0'.[3] Clearly, this point is a point of continuity of f with respect, not merely to the subset PR_0', but with respect to P, and by Baire's theorem f is the limit of a sequence of continuous functions.

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[3] For, a necessary and sufficient condition that $f(x, y)$, defined on an open region R of the xy-plane, be the limit of a sequence of continuous functions is that every set PR_0, where P is a perfect set and R_0 is an open subregion of R, have a point of continuity of f with respect to PR_0. This is a slight modification of Baire's theorem and is proved in a paper *On interval functions* which the author is preparing for publication.