ON A CONVERGENCE THEOREM FOR THE LAGRANGE INTERPOLATION POLYNOMIALS

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The unique polynomial of degree $(n-1)$ assuming the values $f(x_1), \cdots, f(x_n)$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$(1) \qquad L_n(f) = f(x_1)l_1(x) + f(x_2)l_2(x) + \cdots + f(x_n)l_n(x).$$

Here

$$(2) \qquad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n$$

(fundamental polynomials of the Lagrange interpolation), and the polynomial $\omega(x)$ is defined by

$$(3) \qquad \omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n),$$

where c denotes an arbitrary constant not equal to zero. It is known and easy to verify that

$$(4) \qquad l_1(x) + l_2(x) + \cdots + l_n(x) \equiv 1.$$

In the Lagrange interpolation formula let

$$(5) \qquad x_k = x_k^{(n)} = \cos (2k - 1)\pi/2n = \cos \theta_k^{(n)}$$

which implies

$$(6) \qquad \omega(x) = T_n(x) = \cos (n \arccos x) = \cos n\theta, \qquad \cos \theta = x$$

(Tchebyscheff polynomial). In this case we have

$$(7) \qquad l_k(x) = l_k[\theta] = (-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \qquad k = 1, 2, \cdots, n; \; x = \cos \theta;$$

and

$$(8) \qquad L_n(f) = L_n[f; \theta] = \sum_{k=1}^{n} f(\cos \theta_k^{(n)})(-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \qquad x = \cos \theta.$$

Suppose $f(x)$ to be a continuous function; then it is known that the sequence $L_n(f)$, $n=1, 2, \cdots$, is not convergent[1] for all $f(x)$. We may even find a continuous function $f_1(x)$ such that the sequence $L_n(f_1)$, $n=1, 2, \cdots$, is divergent for all points of the interval $-1 \leqq x \leqq +1$.[2]

Therefore it is interesting to prove the following theorem:

THEOREM. *Let $f(x)$ be a continuous function in the interval $-1 \leqq x \leqq +1$; then*

$$\lim_{n\to\infty} \tfrac{1}{2}\{L_n[f; \theta - \pi/2n] + L_n[f; \theta + \pi/2n]\} = f(x), \qquad x = \cos\theta, \tag{9}$$

and the convergence is uniform in the whole interval $-1 \leqq x \leqq +1$.

Between the interpolation polynomials (8) and the partial sums $s_{n-1}(f)$ of the Fourier series of the even function $f(x)$ there is a far reaching analogy. We mention here only the following. On one hand, it is easy to verify that

$$L_n(f) = c_0 + c_1 \cos\theta + \cdots + c_{n-1} \cos (n-1)\theta, \qquad x = \cos\theta, \tag{10}$$

where

$$c_0 = \frac{1}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)}), \qquad c_r = \frac{2}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)}) \cos r\theta_k^{(n)}, \qquad r = 1, 2, \cdots, n-1. \tag{11}$$

On the other hand,

$$s_{n-1}(f) = a_0 + a_1 \cos\theta + \cdots + a_{n-1}\cos (n-1)\theta, \tag{12}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta, \qquad a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta, \qquad r = 1, 2, \cdots, n-1. \tag{13}$$

Our theorem is analogous with the well known theorem of Rogosinski in the theory of Fourier series.

We first prove the following lemma.

[1] G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 190–210. S. Bernstein, *Sur la limitation des valeurs* $\cdots$, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050.

[2] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'intèrpolation*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae, vol. 8 (1937), pp. 131–135.

LEMMA.

$$\frac{1}{2}\sum_{k=1}^{n}\left| l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]\right| < c_1,$$

where $c_1 > 0$ *is an absolute constant.*

From (7) we have, for $\theta \neq \theta_k^{(n)} \pm \pi/2n$,

$$\begin{aligned}
&\tfrac{1}{2}(l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]) \\
&= \tfrac{1}{2}(-1)^{k+1}\left(\frac{\cos n(\theta - \pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta - \pi/2n) - \cos\theta_k^{(n)})} + \frac{\cos n(\theta + \pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta + \pi/2n) - \cos\theta_k^{(n)})}\right) \\
&= \tfrac{1}{2}(-1)^{k+1}\frac{\sin\theta_k^{(n)}\sin n\theta\sin\theta\sin\pi/2n}{4n\sin\frac{1}{2}(\theta + \theta_k^{(n)} + \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)} \\
&\quad\cdot\frac{1}{\sin\frac{1}{2}(\theta + \theta_k^{(n)} - \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\left|\tfrac{1}{2}(l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n])\right| \\
&\leqq \frac{1}{2n}\left|\frac{\sin\pi/2n}{\sin\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)}\right| \\
&\leqq \frac{1}{2n}\,\frac{\pi/2n}{(2/\pi)\left|\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)\right|(2/\pi)\left|\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)\right|} \\
&\leqq \frac{\pi^3}{4n^2}\,\frac{1}{(\theta - \theta_k^{(n)} - \pi/2n)^2}.
\end{aligned} \tag{15}$$

Now let θ be fixed and

$$1 < x_1 < x_2 < \cdots < x_j < x = \cos\theta < x_{j+1} < \cdots < x_n < -1. \tag{16}$$

It is known that[3]

$$|l_k(x)| < 4/\pi, \quad k = 1, 2, \cdots, n;\ n = 1, 2, \cdots;\ -1 \leqq x \leqq +1.$$

Thus

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[3] P. Erdös and G. Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–578. This bound is the best possible; Fejér proved earlier $|l_n(x)| < 2^{1/2}$. See L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55.

$$
(17)\quad
\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{n}\left| l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\right| \\
&\leqq \frac{16}{\pi}+\frac{1}{2}\sum_{1\leqq k\leqq n,\, k\neq j-2,j-1,j,j+1}\left| l_k[\theta-\pi/2n]+l_k[\theta+\pi 2n]\right| \\
&\leqq \frac{16}{\pi}+\sum_{1\leqq k\leqq n,\, k\neq j-2,j-1,j,j+1}\frac{\pi^3}{4n_2}\frac{1}{(\theta-\theta^{(n)}_k-\pi/2n)^2} \\
&\leqq \frac{16}{\pi}+\frac{\pi^3}{4n^2}\sum_{k=1}^{\infty}\left(\frac{2n}{\pi}\right)^2\frac{1}{k^2}=c_1.
\end{aligned}
$$

If $\theta=0,\ \pi$, the same inequality evidently holds.

From (17) we obtain for sufficiently large n, $\delta>0$ fixed,

$$
(18)\quad \sum_{1\leqq k\leqq n,\, |\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}\left| l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\right| = O(1/n).
$$

Now we are in the position to prove our theorem. Let $\epsilon>0$ be a fixed number. The identity (4) gives

$$
(19)\quad \frac{1}{2}\sum_{k=1}^{n}(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]) = 1;
$$

hence for a fixed $x=\cos\theta$

$$
(20)\quad
\begin{aligned}
&\tfrac{1}{2}\{L_n[f;\theta-\pi/2n]+L_n[f;\theta+\pi/2n]\}-f(\cos\theta) \\
&\quad=\sum_{k=1}^{n}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta+\pi/2n]+l_k[\theta+\pi/2n]).
\end{aligned}
$$

The function $f(\cos\theta)$ is continuous; thus we can find a positive number δ such that

$$
(21)\quad \left| f(\cos\theta)-f(\cos\theta_k^{(n)})\right| < \epsilon/2c_1
$$

whenever $\left|\theta-\theta_k^{(n)}\right|<\delta$. From (20) and (21) we have

$$
(22)\quad
\begin{aligned}
\Delta&=\left| \tfrac{1}{2}\{L_n[f;\theta-\pi/2n]+L_n[f;\theta+\pi/2n]\}-f(\cos\theta)\right| \\
&=\Bigg| \sum_{1\leqq k\leqq n,\, |\theta-\theta_k^{(n)}|<\delta}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]) \\
&\quad+\sum_{1\leqq k\leqq n,\, |\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n])\Bigg| \\
&\leqq\frac{\epsilon}{2c_1}\sum_{1\leqq k\leqq n,\, |\theta-\theta_k^{(n)}|<\delta}\tfrac{1}{2}\left| l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\right| \\
&\quad+\sum_{1\leqq k\leqq n,\, |\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}\left| f(\cos\theta_k^{(n)})-f(\cos\theta)\right|\left| l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\right|
\end{aligned}
$$

and by the lemma and (18) for sufficiently large n

$$\Delta \leqq \frac{\epsilon}{2c_1} c_1 + \sum_{1\leqq k\leqq n, |\theta-\theta_k^{(n)}|>\delta} \tfrac{1}{2}2M \mid l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n] \mid$$
$$< \epsilon/2 + MO(1/n) < \epsilon,$$

where $M = \max_{-1\leqq x\leqq +1} |f(x)|$, and this proves our theorem.

BUDAPEST, HUNGARY

DISCONTINUOUS CONVEX SOLUTIONS OF DIFFERENCE EQUATIONS[1]

FRITZ JOHN

This paper contains some conditions for continuity of convex solutions of a difference equation.

A function $f(x)$ defined for $a \leqq x \leqq b$ is *convex*, if

$$(1) \qquad f\left(\frac{x+y}{2}\right) \leqq \frac{f(x) + f(y)}{2} \cdot$$

If $f(x)$ is convex and bounded from above in $a \leqq x \leqq b$, then $f(x)$ is continuous (see Bernstein [1, p. 422]).[2] If $f(x)$ is convex in $a \leqq x \leqq b$ and y a fixed number with $a < y < b$, let the function $\phi_y(x)$ be defined by

$$\phi_y(x) = \lim_{\alpha \to x-y} f(y + \alpha),$$

where α assumes *rational* values only. Then $\phi_y(x)$ is uniquely defined, continuous, and convex for $a < x < b$ (F. Bernstein [1, p. 431, Theorem 7]); moreover $\phi_y(x) = f(x)$ for rational $y - x$.

THEOREM 1. *If there exists at most one continuous convex solution of the difference equation*

$$(2) \qquad F(x, f(x), f(x+1), \cdots, f(x+n)) = g(x), \qquad x > 0,$$

where F and g are continuous functions of their arguments, then there exist no discontinuous convex solutions.

PROOF. If $f(x)$ is a convex solution, then, for $x - y$ rational,

$$F(x, \phi_y(x), \phi_y(x+1), \cdots, \phi_y(x+n)) = g(x);$$

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[1] Presented to the Society, September 12, 1940.

[2] The numbers in brackets refer to the bibliography.