and by the lemma and (18) for sufficiently large n

$$\Delta \leqq \frac{\epsilon}{2c_1} c_1 + \sum_{1 \leqq k \leqq n, |\theta - \theta_k^{(n)}| > \delta} \tfrac{1}{2} 2M \mid l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n] \mid$$
$$< \epsilon/2 + MO(1/n) < \epsilon,$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$, and this proves our theorem.

BUDAPEST, HUNGARY

DISCONTINUOUS CONVEX SOLUTIONS OF DIFFERENCE EQUATIONS[1]

FRITZ JOHN

This paper contains some conditions for continuity of convex solutions of a difference equation.

A function $f(x)$ defined for $a \leqq x \leqq b$ is *convex*, if

$$\left(\frac{x + y}{2}\right) \leqq \frac{f(x) + f(y)}{2}. \tag{1}$$

If $f(x)$ is convex and bounded from above in $a \leqq x \leqq b$, then $f(x)$ is continuous (see Bernstein [**1**, p. 422]).[2] If $f(x)$ is convex in $a \leqq x \leqq b$ and y a fixed number with $a < y < b$, let the function $\phi_y(x)$ be defined by

$$\phi_y(x) = \lim_{\alpha \to x - y} f(y + \alpha),$$

where α assumes *rational* values only. Then $\phi_y(x)$ is uniquely defined, continuous, and convex for $a < x < b$ (F. Bernstein [**1**, p. 431, Theorem 7]); moreover $\phi_y(x) = f(x)$ for rational $y - x$.

THEOREM 1. *If there exists at most one continuous convex solution of the difference equation*

$$F(x, f(x), f(x + 1), \cdots, f(x + n)) = g(x), \qquad x > 0, \tag{2}$$

where F and g are continuous functions of their arguments, then there exist no discontinuous convex solutions.

PROOF. If $f(x)$ is a convex solution, then, for $x - y$ rational,

$$F(x, \phi_y(x), \phi_y(x + 1), \cdots, \phi_y(x + n)) = g(x);$$

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[1] Presented to the Society, September 12, 1940.

[2] The numbers in brackets refer to the bibliography.

as both members of this equation are continuous functions of x, it holds for *all* $x>0$. As there is at most one continuous convex solution, we have

$$\phi_y(x) = \phi_z(x)$$

for all positive z, y, x. As $\phi_z(z)=f(z)$, we see that $f(z)$ is identical with the continuous function $\phi_y(z)$ for all positive z.

THEOREM 2. *If the difference equation* (2) *has at most one solution, which is monotone for sufficiently large* x, *then* (2) *has at most one convex solution, and that solution will be continuous.*

PROOF. Every continuous convex solution is monotone for sufficiently large x. Apply Theorem 1.

THEOREM 3. *A difference equation of the form*

$$\prod_{k=0}^{n} (f(x+k))^{a_k} = g(x), \qquad x > 0, \tag{3}$$

(a_k *real constants*) *has at most one convex solution, if*

(1) *all roots of the equation*

$$\sum_{k=0}^{n} a_k x^k = 0$$

are simple and of absolute value 1,

$$a_n > 0, \qquad \sum_{k=0}^{n} a_k \neq 0,$$

(2) $g(x)\neq 0$ *and continuous*,
(3) $\lim_{x\to\infty} (\log |g(x)|)/x = 0$,
(4) $\lim_{x\to\infty} (\log |g(x)|)/\log x \neq \sum_k a_k$.

PROOF. Assumption (3) implies

$$\prod_{k=0}^{n} | f(x+k) |^{a_k} = | g(x) |.$$

From assumption (1) above and the lemma proved below, it follows that there are *non-negative* constants b_l and c_r, such that

$$\prod_l | g(x+l) |^{b_l} = \prod_l \left[\prod_k | f(x+k+l) |^{a_k} \right]^{b_l}$$
$$= \prod_r | f(x+r) |^{c_r}.$$

For every continuous convex solution $f(x)$ of (3), $|f(x)|$ is either monotone non-decreasing or monotone decreasing for sufficiently large x. Which of these alternatives takes place is determined by $g(x)$, as in the first case

$$\prod_l | g(x + l) |^{b_l}$$

is monotone non-decreasing, and in the second case monotone decreasing for sufficiently large x. If $|f(x)|$ is monotone non-decreasing, we have in $F(x) = \log |f(x)|$ a non-decreasing solution of the equation

$$\sum_k a_k F(x + k) = \log | g(x) |.$$

It follows from our assumptions that such a solution $F(x)$ is uniquely determined for all sufficiently large x, and hence for all x (John [**3**, p. 183]). If $|f(x)|$ is monotone decreasing for sufficiently large x, then $F(x) = -\log |f(x)|$ is an increasing solution of

$$\sum_k a_k F(x + k) = - \log | g(x) |,$$

and hence uniquely determined. Thus for any continuous convex solution $f(x)$ of (3), $|f(x)|$ is uniquely determined. Then $f(x)$ is uniquely determined as well, unless $f(x)$ is linear for sufficiently large x; but if $f(x)$ is linear for large x,

$$\lim_{x\to\infty} \frac{\log | g(x) |}{\log x} = \lim_{x\to\infty} \sum_k a_k \frac{\log |f(x + k)|}{\log x} = \sum_k a_k$$

contrary to assumption. Thus there exists at most one continuous convex solution, and hence at most one convex solution.

Example. The equation

$$f(x + 1) \cdot f(x) = x^{-p}, \qquad x > 0,\ p > 0,$$

satisfies the assumptions of Theorem 3 and hence has at most one convex solution (proved by A. E. Meyer [**4**] for $p = 1$, for general p by H. P. Thielman [**5**]). The convex solution is found to be

$$B(\tfrac{1}{2}x, \tfrac{1}{2})^p/(2\pi)^{p/2}.$$

LEMMA. *If $\phi(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial, such that*

(1) $a_n > 0$,

(2) *$\phi(x)$ has no positive real roots,*

then there are polynomials $\psi(x)$ and $\sigma(x)$ with non-negative coefficients, such that

$$\phi(x) \cdot \psi(x) = \sigma(x).$$

PROOF. $\phi(x)$ can be factored in the form

$$\phi(x) = a_n \prod_l (x + \alpha_l) \prod_k (x^2 + 2\beta_k x + \gamma_k)$$

where $\alpha_l \geqq 0$, $\gamma_k > \beta_k^2$. Hence it is sufficient to prove the lemma for the case that

$$\phi(x) = x^2 + 2\beta x + \gamma$$

and $\beta^2 < \gamma$. We define θ with $0 < \theta < \pi$ by

$$\cos\theta = -\beta/\gamma^{1/2}.$$

Let the non-negative *integer* s be determined by

$$\frac{\pi}{s+2} \leqq \theta < \frac{\pi}{s+1}\cdot$$

Put

$$\psi(x) = \frac{(\gamma^{1/2})^{s+2}\sin(s+1)\theta - x(\gamma^{1/2})^{s+1}\sin(s+2)\theta + x^{s+2}\sin\theta}{[x^2 + 2\beta x + \gamma]\cdot\sin\theta}$$

$$= \sum_{k=0}^{s} (\gamma^{1/2})^k \frac{\sin(k+1)\theta}{\sin\theta} x^{s-k}.$$

$\psi(x)$ and $(x^2+2\beta x+\gamma)\cdot\psi(x)$ obviously are polynomials with non-negative coefficients.

THEOREM 4. *The difference equation*

$$\sum_{k=0}^{n} a_k f(x+k) = g(x), \qquad x > 0, \tag{4}$$

has no discontinuous convex solutions, if

(a) *$g(x)$ is bounded from above in every positive interval,*

(b) $a_n > 0$,

(c) *the equation $\sum_{k=0}^{n} a_k x^k = 0$ has no positive real roots.*

PROOF. Let $\phi(x) = \sum_{k=0}^{n} a_k x^k$. Let $\psi(x)$ be a polynomial, such that $\psi(x)$ and $\phi(x)\cdot\psi(x)$ have no negative coefficients. Let

$$\phi(x)\cdot\psi(x) = x^s\cdot\sigma(x),$$

where $\sigma(x)$ is a polynomial of degree m with $\sigma(0) \neq 0$. Then

$$\sigma(x)\cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_{k=0}^{2m} c_k x^k$$

is a polynomial of degree $2m$ with $c_k = c_{2m-k} \geqq 0$. Put

$$\tau(x) = \psi(x)\cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_r b_r x^r.$$

Then

$$\phi(x)\cdot\tau(x) = x^s \sum_{k=0}^{2m} c_k x^k,$$

where $\tau(x)$ is a polynomial with non-negative coefficients b_r and $c_k = c_{2m-k} \geqq 0$. For a convex solution $f(x)$ of (4)

$$\begin{aligned}\sum_r b_r g(x+r) &= \sum_{k=0}^{2m} c_k f(x+k+s)\\ &= \frac{1}{2}\sum_{k=0}^{2m} c_k [f(x+k+s) + f(x+2m-k+s)]\\ &\geqq \frac{1}{2}\sum_{k=0}^{2m} c_k f(x+s+m),\end{aligned}$$

or

$$f(x+s+m) \leqq \frac{2\sum_r b_r g(x+r)}{\sum_k c_k}.$$

As $g(x)$ is bounded above, it follows that $f(x)$ is bounded above, and hence continuous.

THEOREM 5. *If the difference equation*

$$\sum_{0=k}^{n} a_k f(x+k) = g(x) \tag{5}$$

has a continuous convex solution, and if the equation $\sum_{k=0}^{n} a_k x^k = 0$ *has a positive real root, then the difference equation has discontinuous convex solutions as well.*

PROOF. It is sufficient to prove that the equation

$$\sum_{k=0}^{n} a_k f(x+k) = 0$$

has a discontinuous convex solution, as the sum of two convex functions is again convex. Let

$$\sum_{k=0}^{n} a_k x^k = (x-\lambda)\sum_{k=0}^{n-1} b_k x^k,$$

where $\lambda > 0$. Then

$$\sum_k a_k f(x+k) = \sum_k b_k[f(x+k+1) - \lambda f(x+k)].$$

It is sufficient to show that the equation

$$f(x+1) - \lambda f(x) = 0 \tag{6}$$

has a discontinuous convex solution.

Let Σ be a basis for all real numbers; that is, every real x may be represented in one and only one way in the form

$$x = \alpha a + \beta b + \cdots + \gamma c,$$

where $a, b, \cdots, c$ are in Σ, and $\alpha, \beta, \cdots, \gamma$ are rational numbers. (The existence of such a base is proved by Hamel [**2**].) Without restriction of generality we may assume 1 to be an element of Σ (this comes back to assuming a normal ordering of the set of real numbers with 1 as first element). For every real x, there is then uniquely determined a number α, such that

$$x = \alpha\cdot 1 + \beta b + \cdots + \gamma c,$$

where $\alpha, \beta, \cdots, \gamma$ are rational, and $1, b, \cdots, c$ are in Σ. If $\lambda \neq 1$, define $f(x)$ by λ^α. Then

$$f(x+1) - \lambda f(x) = \lambda^{\alpha+1} - \lambda\cdot\lambda^\alpha = 0.$$

If $y = \alpha' 1 + \beta' b + \cdots + \gamma' c$ is the representation of y then

$$f\left(\frac{x+y}{2}\right) = (\lambda^\alpha\cdot\lambda^{\alpha'})^{1/2} \leqq \frac{\lambda^\alpha + \lambda^{\alpha'}}{2} = \frac{f(x)+f(y)}{2}.$$

Hence $f(x)$ is a convex solution of (6). $f(x)$ is discontinuous in every interval, as for an element $b \neq 1$ of Σ

$$f(\beta b) = 1$$

for all rational β, whereas

$$f(\alpha) = \lambda^\alpha \neq 1$$

for all rational $\alpha \neq 0$. Similarly, if $\lambda = 1$, we define $f(x) = \beta$, where β is the coefficient of the fixed element $b \neq 1$ of Σ. Then

$$f(x+1) - f(x) \equiv 0, \qquad f\left(\frac{x+y}{2}\right) \equiv \frac{f(x)+f(y)}{2},$$

$f(\alpha) = 0$ for all rational α, but $f(\beta b) = \beta \neq 0$ for all rational $\beta \neq 0$.

Note added March 15: I am indebted to Professors O. Szász and G. Szegö for the information that the Lemma had been proved by E. Meissner, Mathematische Annalen, vol. 70 (1911), pp. 223–235. See also Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 730, no. 190.

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