MOMENT PROBLEM FOR A BOUNDED REGION[1]

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1. **Introduction.** In this paper a solution of the moment problem given by Hausdorff[2] for a bounded interval is extended to any bounded region in euclidean n-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the n-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the "class" of Fourier and Fourier-Stieltjes series.[3]

2. **Definitions and notation.** Let n be a positive integer, fixed but arbitrary. R^n will denote the euclidean n-space, (x) and (y) will stand for $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, points of R^n, and E a bounded, closed subset of R^n. ν, τ, i, j, k, and s, will be used for non-negative integers, and (k), (s), and so on, will denote ordered n-tuples of non-negative integers $(k_1, k_2, \cdots, k_n)$, $(s_1, s_2, \cdots, s_n)$, and so on, $(k)=(s)$ will mean $k_i = s_i$, $i=1, 2, \cdots, n$. (0) will mean $(0, 0, \cdots, 0)$, $\{\mu_{(m)}\}$ will be a sequence of real numbers, and $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ will be two sequences of polynomials such that

$$U_{(0)}(x) = V_{(0)}(x) = \text{const.},$$

$$\int_E U_{(k)}(x)V_{(s)}(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases} \tag{1}$$

and by $\int_{E(y)} f(x, y)d\Phi(E)$ will be meant the Lebesgue-Stieltjes integral over E of f considered as a function of a point (y). B will be used for any Borel set with $B \subseteq E$.

If f is integrable over E we define

$$\mathfrak{S}(f, x) \simeq \sum_{(k)} A_{(k)}V_{(k)}(x), \qquad A_{(k)} = \int_E f(x)U_{(k)}(x)dx,$$

$$S(x, y) \simeq \sum_{(k)} U_{(k)}(x)V_{(k)}(y).$$

Let L_ν for every ν be a partition of R^n into two subsets, one closed and bounded. We write $(k) \in L_\nu$ to indicate that (k) belongs to the

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[2] F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), pp. 220–248.

[3] See, for example, A. Zygmund, *Trigonometrical Series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86.

bounded subset defined by L_ν, and require that for every (k) there exist a ν such that $(k)\in L_\nu$, and that $(k)\in L_\nu$ shall imply $(k)\in L_{\nu'}$ for all $\nu'\geqq\nu$. Now let

$$S_\nu(x, y) = \sum_{(k)\,\epsilon\, L_\nu} U_{(k)}(x)V_{(k)}(y),$$

$$\mathfrak{S}_\nu(f, x) = \sum_{(k)\,\epsilon\, L_\nu} A_{(k)}V_{(k)}(x) = \int_E S_\nu(x, y)f(y)dy.$$

If $T: \|a_{ij}\|$ is any regular Toeplitz transformation,[4] we write

$$T\mathfrak{S}_\nu(f, x) = \int_E TS_\nu(x, y)f(y)dy = \int_E K_\nu(x, y)f(y)dy.$$

If P is a polynomial in (x) we denote by $\mu_{(x)}(P)$ the expression resulting from the substitution of $\mu_{m_1,m_2,\cdots,m_n}$ for $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in P.

3. **Moment problem.** A solution of the moment problem for the set E is given in the following theorem:

THEOREM. *Given $\{U_{(k)}(x)\}$, $\{V_{(k)}(x)\}$, $\{L_\nu\}$, and T satisfying the conditions above, and such that $TS_\nu(x, y)=K_\nu(x, y)\geqq 0$ for all (x), $(y)\in E$, and all ν, and such that for any f integrable over E $T\mathfrak{S}_\nu(f, x)\to f(x)$ for every $(x)\in E$ for which f is continuous, and uniformly on E if f is continuous on E, then in order that a sequence $\{\mu_{(m)}\}$ be expressible in the form*

$$\mu_{m_1,m_2,\cdots,m_n} = \int_E x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}d\Phi(E),$$

where Φ is completely additive, defined over at least all Borel sets of R^n, and with

(1) $\int_E|d\Phi(E)|\leqq M$,
(2) $\Phi(B)\geqq 0$,
(3) $\Phi(B)=\int_B\phi(x)dx$ *and with*
(3a) $\phi\in L_E^p$, $p>1$,
(3b) $\phi\in L_E$,
(3c) $|\phi|\leqq M$,
(3d) $\phi\in C_E$,

it is necessary and sufficient that

(1) $\int_E|\mu_{(y)}\{K_\nu(x, y)\}|dx\leqq M$ *for all* ν,
(2) $\mu_{(y)}\{K_\nu(x, y)\}\geqq 0$ *for all* $(x)\in E$ *and all* ν,
(3a) $\int_E|\mu_{(y)}\{K_\nu(x, y)\}|^p dx\leqq M$ *for all* ν,

[4] Zygmund, loc. cit., pp. 40–43.

(3b) $\lim_{\nu,\tau\to\infty}\int_E|\mu_{(y)}\{K_\nu(x, y)\}-\mu_{(y)}\{K_\tau(x, y)\}|dx=0$,
(3c) $|\mu_{(y)}\{K_\nu(x, y)\}|\leqq M$ *for all* $(x)\in E$ *and all* ν,
(3d) $\lim_{\nu,\tau\to\infty}|\mu_{(y)}\{K_\nu(x, y)\}-\mu_{(y)}\{K_\tau(x, y)\}|=0$ *uniformly in* $(x)\in E$.

The proof in each of the six cases closely parallels that of Hausdorff.[2] The proof is given for case (1) to indicate the modifications:

Necessity. We have

$$|\mu_{(y)}\{K_\nu(x, y)\}| = \left|\int_{E(y)} K_\nu(x, y)d\Phi(E)\right| \leqq \int_{E(y)} K_\nu(x, y)\,|d\Phi(E)|,$$

$$\int_E |\mu_{(y)}\{K_\nu(x, y)\}|\,dx \leqq \int_E\left\{\int_E K_\nu(x, y)dx\right\}|d\Phi(E)| \leqq C\int_E |d\Phi(E)| \leqq M$$

for

$$K_\nu(x, y) = \sum_{j=0}^{\infty} a_{\nu j}\sum_{(k)\in L j} U_{(k)}(x)V_{(k)}(y)$$

$$\int_E K_\nu(x, y)dx = \sum_{j=0}^{\infty} a_{\nu j}\sum_{(k)\in L j} V_{(k)}(y)\int_E U_{(k)}(x)dx = \sum_{j=0}^{\infty} a_{\nu j} \leqq \sum_{j=0}^{\infty}|a_{\nu j}| \leqq C.$$

Sufficiency. Let

$$\Phi_\nu(B) = \int_B \mu_{(y)}\{K_\nu(x, y)\}dx,$$

$$\int_E |d\Phi_\nu(E)| = \int_E |\mu_{(y)}\{K_\nu(x, y)\}|\,dx \leqq M$$

and, by a well known theorem of Helly, there is a subsequence $\{\Phi_{\nu'}\}$ and a function Φ such that $\int_E|d\Phi(E)|\leqq M$ and $\Phi_{\nu'}(B)\to\Phi(B)$, and also $\int_E V_{(k)}(y)d\Phi_{\nu'}(E)\to\int_E V_{(k)}(y)d\Phi(E)$ whence $\mu_{(y)}\{V_{(k)}(y)\}=\int_E V_{(k)}(y)d\Phi(E)$, and Φ is a solution.

4. **Examples and conclusion.** If E is the unit sphere in R^n, $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ may be taken as the normalized polynomials of Appell-

Didon,[5] $(k)\in L_\nu$ to mean $\sum_{i=1}^{n} k_i \leqq \nu$, and T any (C, r) with $r \geqq n+1$.[6] In particular, for $n=1$ this reduces to the Hausdorff solution for the unit interval. If E is the circumference of the unit circle we may set $U_0(x)=V_0(x)=(2\pi)^{-1/2}$, and, for $k>0$,

$$U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2} \cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2} \sin k\theta$$

with $(s)\in L_\nu$ meaning $s \leqq 2\nu$, T any (C, r) with $r \geqq 1$.[7] Sequences $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ can be constructed by the Schmidt process for any bounded region in R^n. It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

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[5] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynomes d'Hermite*, Paris, 1926.

[6] L. Koschmieder, *Über die C-Summierbarkeit gewisser Reihen von Didon und Appell*, Mathematische Annalen, vol. 104 (1931), pp. 387–402.

[7] L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.