RECURRENCE OF SYMBOLIC ELEMENTS IN DYNAMICS[1]

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1. **Introduction.** Morse and Hedlund [1] have given a symbolic treatment of modern theoretical dynamics as developed by Birkhoff [2] and others. In the Morse-Hedlund viewpoint the concept of recurrence plays an important role. To establish various theorems on symbolic trajectories Morse and Hedlund introduced "symbolic elements," the analogues of line elements on ordinary trajectories, a symbolic element being the notion of a trajectory T and a particular symbol in T. In the present paper we shall be concerned primarily with the question: "*How are recurrence of a trajectory T and elements based on T related?*"

2. **Definitions.** For terms defined elsewhere and used here the reader is referred to other papers [**1**, **3**]. Let rays R_i ($i=1, 2, 3, 4$) be given by

$$\alpha_{i1}\ \alpha_{i2}\ \alpha_{i3} \cdots .$$

The distance E_1E_2 between the elements $E_1=(R_1, R_2)$ and $E_2=(R_3, R_4)$ is defined to be $1/n$ where n is such that $a_{1j}=a_{3j}$, $a_{2j}=a_{4j}$ for each value of j in the range $1, 2, \cdots, n$, while

$$(a_{1,n+1},\ a_{2,n+1}) \neq (a_{3,n+1},\ a_{4,n+1}).$$

The element E_1 is the ray

$$A_1A_2A_3 \cdots , \tag{1}$$

where A_j denotes the pair of symbols (a_{1j}, a_{2j}). In what follows the term "recurrence of E_1" designates the recurrence of the ray (1). We recall that a ray (1) is *recurrent* if for each n there is an m such that each n-block in (1) is contained in each m-block of (1). For each n the value $R(n)$ of the *recurrency* function of (1) is the smallest m with the property just mentioned.

3. **Recurrence.** It is evident that the recurrence of a single element based on a trajectory T does not imply the recurrence of T.

THEOREM 1. *If each element based on a trajectory T is recurrent, T is recurrent.*

The recurrence of each ray based on T obviously implies the recur-

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rence of T. The recurrence of each element based on T implies the recurrence of each ray based on T, whence T is recurrent.

THEOREM 2. *There exists a recurrent trajectory with a non-recurrent element.*

If α is irrational, the trajectory $T(0, \alpha)$ of Morse and Hedlund [4] is recurrent and is of the form $R^{-1}abR$. The element (bR, aR) is not recurrent.

The proof of the theorem to follow is simple and will be omitted.

THEOREM 3. *A trajectory T is identical with its inverse if and only if $T = R^{-1}R$ or $T = R^{-1}aR$ for some ray R and symbol a.*

THEOREM 4. *If a trajectory T is recurrent and is identical with its inverse, each element based on T is recurrent.*

We assume first that $T = R^{-1}R$ for some ray R. Consider an element $E = (B^{-1}R, R_1)$, where B is such that $R = BR_1$. We permit B to be vacuous, in which case $E = (R, R_1)$. Let G denote the leading s-block of R_1 for a given value of s, and let r denote the length of B. Let J denote the leading s-block of $B^{-1}R$. Then J in $B^{-1}R$ matches G in R_1, and the pair of blocks J, G corresponds to the block $A_1 \cdots A_s$ in the representation (1) of E. The block $G^{-1}B^{-1}BG$ is a subblock of T. Let $R(n)$ denote the recurrency function of T, whence $G^{-1}B^{-1}BG$ occurs in each $R(2r+2s)$-block of T. Let n be the number of symbols preceding one such block $G^{-1}B^{-1}BG$ in R, so that

$$R = HG^{-1}B^{-1}BGR_2,$$

where H is a block of length n and R_2 is a subray of R. For brevity we introduce the notation $K = G^{-1}B^{-1}BG$ so that $R = HKR_2$. It is no restriction on the generality of the argument to follow to suppose that $n \geqq r$ so that the block K is a subblock of R_1. The final block G in K is preceded in R_1 by a subblock of length $n+r+s$. In the ray $B^{-1}R$ the final subblock $B^{-1}BG$ of K is preceded by a block of length $n+r+s$. Thus the initial s-block J of the subblock $B^{-1}BG$ of K in $B^{-1}R$ matches the final block G of K in R_1. We have proved that in each $R(2r+2s)$-block of R_1 there is a subblock G matching a subblock J of $B^{-1}R$ whence each $R(2r+2s)$-block in the representation (1) contains a block identical with $A_1 \cdots A_s$. Since each block in (1) is contained in the block $A_1 \cdots A_s$ for large enough s, it follows that E is recurrent.

We suppose, finally, that $T = R^{-1}aR$ for some ray R and symbol a. Let r and s be arbitrary non-negative integers. We consider an element $E=(D^{-1}aR, R_1)$ where D is the leading r-block of R so that $R=DR_1$, R_1 being a subray of R. We let G denote the leading s-block of R_1 so that $R_1=GR_2$ where R_2 is a subray of R_1. Evidently, T contains the subblock $G^{-1}D^{-1}aDG$ and the initial block G of R_1 matches the leading s-block J of $D^{-1}aR$. The last block G in any block $G^{-1}D^{-1}aDG$ of R_1 matches the leading block J of a subblock $D^{-1}aDG$ in $D^{-1}aR$, whence E is recurrent.

By Theorem 3 the above argument proves that each element E based on T is recurrent.

THEOREM 5. *If for some block C and ray R a trajectory T is of the form $R^{-1}CR$, and if each element based on T is recurrent, the trajectory T is identical with its inverse.*

We suppose first that C is of even length $2q$, and we accordingly write $C=C_1C_2$ where C_1 and C_2 are of length q. We consider the element $E=(C_1^{-1}R, C_2R)$ based on T. Since E is recurrent, C_2 occurs in the subray R matched in C_2R by a block C_1^{-1} in $C_1^{-1}R$. It follows that C_1^{-1} matches C_2 in R, whence $C_1=C_2^{-1}$ and $T=T^{-1}$.

If now C is of odd length $2q+1$, we write $C=C_1aC_3$ where C_1 and C_3 are of length q. Let the initial symbol of R be denoted by b. Since the element $E=(C_1^{-1}R, aC_3R)$ is recurrent, there is a subblock aC_3 of R matched in aC_3R by a block $C_1^{-1}b$ in the ray $C_1^{-1}R$. Hence there is a block C_3 in R matching a block C_1^{-1} in R. Thus $C_1=C_3^{-1}$ and $T=T^{-1}$.

The Morse recurrent trajectory T extensively studied in the literature is a trajectory identical with its inverse, whence each element based on T is recurrent. Morse and Hedlund gave methods termed projection, reduction, association, and substitution by which one can obtain a recurrent trajectory from a given recurrent trajectory. One can prove readily that the elements based on the trajectories obtained by derivation from the Morse recurrent trajectory are recurrent.

Elsewhere the author proved (3) that the exponent trajectory T_e of a recurrent trajectory T is generated by a finite number of symbols, and is recurrent. Let $T_1, T_2, \cdots$ be a sequence of trajectories such that T_i is the exponent trajectory of T_{i-1} for each i. We term $T_2, T_3, \cdots$ the *successive exponent trajectories* of T. One can prove readily that the successive exponent trajectories of the Morse recurrent trajectory are each of the form $R^{-1}aR$ where a is a symbol and R is a ray, whence the elements based on these successive trajectories are recurrent.

We let $R_i(n)$ denote the recurrency function of the element E_i based on a trajectory T. If for each i the set $R_i(1), R_i(2), \cdots$ is bounded, T is said to be *E-recurrent*. For the Morse recurrent trajectory one can prove that the set $R_i(1)$ is not bounded whence this trajectory is not E-recurrent. The existence of a nonperiodic E-recurrent trajectory is still an open question.

References

1. M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics, vol. 60 (1938), pp. 815–866.
2. G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9, New York, 1927.
3. R. Oldenburger, *Exponent trajectories in symbolic dynamics*, Transactions of this Society, vol. 46 (1939), pp. 453–466.
4. Morse and Hedlund, *Symbolic dynamics*, II. *Sturmian trajectories*, American Journal of Mathematics, vol. 52 (1940), pp. 1–42.

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