233. H. Schwerdtfeger: *A complete parametrization of the symplectic group.*

While all known parametrizations of the symplectic group omit certain "exceptional" elements, the parametrization derived in the present paper covers the whole group. Let ϵ be the n-rowed unit matrix and F the matrix with first row 0,0 and second row ϵ,0. A $2n$-rowed real regular matrix T is proved to be symplectic if and only if $T'FT-F=H$ is symmetric (contact condition), and satisfies the condition: $(F'-F)\cdot(H+F)$ is idempotent. To prove the sufficiency of the latter condition one has to show that for any such matrix H a symplectic T can be found with which it is associated by the said relation. By establishing a set of normal forms for H under the transformation $S'HS$ where S is a $2n$-rowed matrix for which $S'FS=F$, that is, S is the matrix with first row σ,0 and second row $0,\sigma'^{-1}$, a set of normal matrices T can be found such that RTS is the most general symplectic matrix where R is the matrix with first row ρ,0; second row $0,\rho'^{-1}$ and ρ,σ are any regular n-rowed matrices. The method has been carried through in detail for $n=2$. (Received May 1, 1942.)

Analysis

234. Einar Hille: *Notes on linear transformations.* IV. *Representation of semi-groups.*

Let $\{T_s\}$ be a semi-group of linear bounded transformations on a separable Banach space to itself, defined for $s>0$. Let T_s be weakly measurable and $\|T_s\|\leqq 1$ for $s>0$. Let T_s (E) be dense in E. Put $A_h=(1/h)[T_h-I]$. For $h\to 0$, $T_hx\to x$ everywhere in E and $A_hx\to Ax$ in a dense set $D(A)$. Here A is linear, closed and ordinarily unbounded. The resolvent $R(\lambda)$ of A is the negative of the Laplace transform of T_s and is bounded for $\Re(\lambda)>0$. Conversely, T_s is expressible in terms of $R(\lambda)$ by the inversion formula for Laplace integrals which gives an interpretation of T_s as exp (sA). A further interpretation is given by $T_sx=\lim_{h\to 0}\exp[sA_h]x$, valid in $D(A)$. The method is essentially that of Stone. (Received April 6, 1942.)

235. Witold Hurewicz: *An ergodic theorem without invariant measure.*

Let E be an abstract space carrying a completely additive measure μ defined on a completely additive field Ω of subsets of E (it is assumed that a set $X\in\Omega$ with $\mu(X)=\infty$ can always be split into a countable number of sets with finite measures). Let T be a one-to-one point transformation of E on itself satisfying the conditions: (1) $X\in\Omega$ implies $T(X)\in\Omega$; (2) $T(X)\subset X\in\Omega$ implies $\mu(X-T(X))=0$ (incompressibility condition). Finally let $F(X)$ $(X\in\Omega)$ be an additive finite-valued set function, absolutely continuous with respect to the measure μ. For $X\in\Omega$, let $F_n(X)=\sum_{i=0}^{n}F(T^i(X))$, $\mu_n(X)=\sum_{i=0}^{n}\mu(T^i(X))(n=0,1,2,\cdots)$. Since F_n is absolutely continuous with respect to the measure μ_n, $F_n(X)=\int_x f_n d\mu_n$, where f_n is a measurable point function defined on E. It can be shown that the sequence $\{f_n\}$ converges almost everywhere to a function ϕ, invariant with respect to T. By "almost everywhere" is meant that the points of divergence form a set M such that $\mu(T^n(M))=0$ for $n=1,2,\cdots$. If the measure μ is finite and invariant with respect to T, this theorem coincides with Birkhoff's ergodic theorem. (Received April 24, 1942.)

236. William Karush: *A sufficiency theorem for an isoperimetric problem in parametric form with general end conditions.*

The problem studied is that of minimizing an integral $I(C)=g(a)+\int_{t_1}^{t_2}f(a,y,y')dt$

in a class of admissible arcs C: a_h, $y_i(t)$ $(h=1, 2, \cdots, r; i=1, 2, \cdots, n; t_1 \leqq t \leqq t_2)$ in ay-space satisfying conditions of the form $y_i(t_s)=y_{is}(a)$ and $I_\sigma(C)=g_\sigma+\int_{t_1}^{t_2} f_\sigma dt=0$ $(s=1, 2; \sigma=1, 2, \cdots, m)$. The parameters a_h are independent of t. It is shown that a set of sufficient conditions for a proper strong relative minimum on an arc E which does not intersect itself is that E satisfy the multiplier rule including the transversality conditions, nonsingularity, the strengthened Weierstrass condition, and that the second variation of I along E be positive. The method of proof is a generalization of methods used by M. R. Hestenes for non-parametric problems. In particular, use is made of a generalized field theory and of properties of broken extremals. Among other things it is shown that in the sufficiency proof one can assume, without loss of generality, that E is an extremal of no integral of the form $c_\sigma I_\sigma$ where the constants c_σ are not all zero, that is, that E is normal relative to the fixed end point case. (Received April 17, 1942.)

237. Walter Leighton and W. J. Thron: *On the convergence of continued fractions to meromorphic functions.*

In the continued fraction $1+K(a_n x/1)$ let the numbers a_n be complex and denote by L (the point ∞ may be in this set) the set of the limit points of the sequence $\{a_{2n+1}\}$. If $\lim a_{2n}=0$, the continued fraction converges to a meromorphic function of the complex variable x in every region contained in the set D, where D is defined as follows: $x \in D$ if $|x+1/k| > \epsilon$ for all $k \in L$, $k \neq 0$, and if $|x| < M$. The constants ϵ and M are arbitrary. In this theorem the role of the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ can be interchanged. (Received May 28, 1942.)

238. A. N. Lowan and H. E. Salzer: *Coefficients in formulae for numerical integration.*

The values of $B_n^{(n)}(1)/n!$ and $B_n^{(n)}/n!$, where $B_n^{(n)}(1)$ denotes the nth Bernoulli polynomial of the nth order for $x=1$ and $B_n^{(n)}$ denotes the nth Bernoulli number of the nth order, were computed for $n=1, 2, \cdots, 20$. The quantities $B_n^{(n)}(1)/n!$ are required in the Laplace formula of numerical integration employing forward differences, as well as in the Gregory formula and the central difference formulae. The quantities $B_n^{(n)}/n!$ are used in the Laplace formula employing backward differences. (See Milne-Thomson, *Calculus of Finite Differences*, pp. 181–187, 191–193.) The above results were obtained in the course of work carried out by Mathematical Tables Project (Work Projects Administration, New York City), on computational tools for numerical integration. (Received May 21, 1942.)

239. Szolem Mandelbrojt and F. E. Ulrich: *On a generalization of the problem of quasi-analyticity.*

If $\{M_n\}$ is a sequence of positive numbers, let $C(M_n)$ be the class of functions infinitely differentiable on $0 \leqq x < \infty$ such that to each function $f(x)$ of the class there corresponds a positive constant k with the property that $|f^{(n)}(x)| < k^n M_n$ $(n \geqq 1)$ for all $0 \leqq x < \infty$. Let (1) $T(r)=\text{l.u.b.}(r^n/M_n)$ for $n \geqq 1$. It is known that if a function $f(x)$ belongs to a class $C(M_n)$ such that (2) $\int_1^\infty \log T(r)dr/r^2=\infty$, and if $f^{(n)}(0)=0$ $(n \geqq 0)$, then $f(x) \equiv 0$. In the present paper conditions are obtained under which a function belonging to $C(M_n)$ will be identically zero when it is supposed that only a partial subsequence of its derivatives are zero at $x=0$. For instance, suppose that $f(x)$ belongs to $C(M_n)$, (3) $\int_0^\infty |f(x)|dx < \infty$ and (4) $f^{(\nu_n)}(0)=0$ $(n \geqq 1)$ with (5) $G=\limsup_{n\to\infty}(\nu_n/n) < 2$. If (6) $\limsup_{n\to\infty}|f^{(n)}(0)|^{1/n} < \infty$ and if (2) holds, then $f(x) \equiv 0$. Condition (6) can be removed and the conclusion will still be valid if in (2) $T(r)$ is replaced by $T_\omega(r) =$ l.u.b. $(r^{\omega\lambda_n}/M_{\lambda_n+1})$ for $n \geqq 1$, where $\{\lambda_n\}$ is the sequence of positive integers complementary to the sequence $\{\nu_n\}$ with respect to the non-negative integers, and ω is a positive constant less than $(2/G)-1$. If $f(x)$ is even and periodic, condition (3) can be removed. (Received May 11, 1942.)

240. Marston Morse: *Unstable minimal surfaces bounded by a rectifiable contour.*

Recent theorems of Morse and Tompkins and of Shiffman establish the existence of an unstable minimal surface of disc type under appropriate hypothesis on the minimizing sets of the Douglas Dirichlet integral, and on the bounding curve g. A similar theorem is now proved by the author supposing g is no more than simple and rectifiable, and that at least two relative disjoint minimizing sets belong to the area functional A. The surfaces admitted are harmonic and of disc type. The method of the author does not require the analysis of the harmonic surfaces bounded by polygons as does the method of Courant and Shiffman (announced orally recently but not yet published). The theorem is a corollary of theorems proved by Morse and Tompkins in their recent paper in the Annals of Mathematics on *Minimal surfaces of non-minimum type by a new mode of approximation.* The theorem of Morse and Tompkins on the relative continuity of A covers the most difficult point in the analysis. (Received April 17, 1942.)

241. M. E. Munroe: *On the finite dimensionality of certain Banach spaces.*

It is shown that if B is a Banach space, a necessary and sufficient condition that every unconditionally convergent series of elements of B be absolutely convergent is that B be finite-dimensional. On the basis of this result it is shown that finite dimensionality of the range space is necessary and sufficient that (1) all additive absolutely continuous functions of measurable sets be of bounded variation, (2) all Pettis integrals be of bounded variation, (3) all additive absolutely continuous functions of measurable sets be Bochner-Dunford integrals, and (4) the Pettis and Bochner-Dunford integrals be equivalent. (Received May 13, 1942.)

242. Mary De Pazzi Rochford: *Completely non-integrable pfaffians.*

The pfaffian $\sum P_s dx_i$ is called completely non-integrable if there does not exist even a continuous function λ which is an integrating factor, that is, makes $\int \sum P_i dx_i$ independent of the path. If (1) $P_i(x_1, \cdots, x_n)$ is continuous for $i=1, \cdots, n-1$, (2) $P_n=0$, (3) P_1 actually depends upon x_n, that is, $P_1(x_1, \cdots, x_{n-1}, x_n') \neq P_1(x_1, \cdots, x_{n-1}, x_n'')$ for a system $x_1, \cdots, x_{n-1}$, (4) $P_2, \cdots, P_{n-1}$ do not depend upon x_n, then $P_1dx_1 + \cdots + P_{n-1}dx_{n-1}$ is completely non-integrable. (Received May 20, 1942.)

243. Mary De Pazzi Rochford: *Differentiability properties of certain functions of two variables.*

Let P and Q be two continuous functions of x,y. If $\int Pdx+Qdy$ is independent of the path, then for each y_0 and y_1 there exists a function $\eta(x,y_0,y_1)$ such that $Q(x, \eta(x,y_0,y_1))$ has a total derivative with respect to x which is equal to the difference quotient $[P(x,y_0)-P(x,y_1)]/(y_0-y_1)$. For each continuous function g of one variable, even if g is nowhere differentiable, there exists a function $\eta(x,y_0,y_1)$ such that $g(x + \eta(x,y_0,y_1))$ has a derivative with respect to x which is equal to $[g(x+y_0)-g(x$

$+y_1)]/(y_0-y_1)$. If $\partial(\int Qdy)/\partial x$ exists, then each integrating factor of $Pdx+Qdy=0$ has a similar differentiability property. (Received May 20, 1942.)

244. J. D. Rommel: *On conservative transformations of functions of two variables.*

Necessary and sufficient conditions are given in order that $y(s, t)=\int_0^\infty\int_0^\infty k(s, t, \sigma, \tau) x(\sigma, \tau)d\sigma d\tau$ shall transform $x(\sigma, \tau)$ such that $\lim_{\sigma,\tau\to\infty} x(\sigma, \tau)$ exists into $y(s, t)$ such that $\lim_{s,t} y(s, t)$ exists. From this result a characterization of regularity is obtained. By considering step-kernels and step-functions, regular transformations of double sequences are characterized. (Received April 18, 1942.)

245. Raphael Salem and D. C. Spencer: *The influence of gaps on density of integers*

An infinite sequence of integers is said to have the complete gap property with respect to $\omega(x)$ ($\omega(x)$ being a positive non-decreasing function for $x\geqq 0$) if in every closed interval $(a,a+l)$ $(a\geqq 0,\ l\geqq l_0)$ there exists an open interval of length not less than $\omega(l)$ which contains no point of the sequence. Such a sequence will be denoted by $S[\omega(x)]$. Let $\nu(m)$ be the number of terms of the sequence not greater than m. The following theorems are proved: (1) If the integral $\int_0^\infty(\omega(x)/x^2)dx$ diverges, $\lim \nu(m)/m=0$. (2) If $\omega(x)$ is such that $\omega(x)/x^2$ decreases and the integral converges, there exists an $S[\omega(x)]$ with $\liminf \nu(m)/m>0$. (3) If $\omega(x)=\theta x$ $(0>\theta>1/3)$, then $\nu(m)/m=O(m^{-\alpha})$ where $\alpha=[\log (1-2\theta)/(1-3\theta)]/[\log 2(1-2\theta)/(1-3\theta)]$. (4) If $0<\theta<1/3$, there exists a sequence $S[\theta x]$ with $m^{-\alpha}=O(\nu(m)/m)$. (5) If $\omega(x)=\theta x$ $(1/3\leqq\theta<1/2)$, then $\nu(m)=O(\log m)$. (6) If $1/3\leqq\theta<1/2$, there exists a sequence $S[\theta x]$ with $\log m=O(\nu(m))$. (7) If $\omega(x)\geqq x/2$, there exists no infinite sequence $S[\omega(x)]$. (Received April 30, 1942.)

Applied Mathematics

246. H. K. Brown: *The resolution of boundary value problems by means of the finite Fourier transformation.*

The finite sine transformation and the finite cosine transformation are defined as the linear functional operations $\mathfrak{S}\{G\}=\int_0^\pi G(x)\sin nxdx=g_s(n)$ and $\mathfrak{C}\{G\}=\int_0^\pi G(x)\cdot\cos nxdx=g_c(n)$, respectively. The inversion of the product of the transforms of G and H can be made by means of four Faltung theorems. The finite sine transformation was applied to a problem in general heat flow in one dimension in which the nonhomogeneous linear partial differential equation has coefficients which may be functions of the time. It was proved in detail that this problem can be resolved into a standard heat flow problem which has a differential equation of simpler type and constant coefficients. The Faltung theorems permitted an inversion in closed form of the solution of the transformed boundary value problem. A general heat flow problem of similar type in three dimensions was resolved into the same standard problem in one dimension. By the introduction of a fundamental set of solutions of the transformed problem it was possible to make a resolution of the boundary value problems of a general vibrating string, membrane and beam. (Received April 11, 1942.)

Geometry

247. C. J. Everett: *Affine geometry of vector spaces over rings.*

Methods of E. Artin's *Coordinates in affine geometry* (Publication of the University of Notre Dame, Reports of a Mathematical Colloquium, (2), Issue 2) are used to show