ON A THEOREM OF NEWSOM

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1. **Extension of the theorem.** In 1938, Newsom[1] published a paper containing a theorem regarding the behavior for large values of $|z|$ of the function

$$f(z) = \sum_{n=0}^{\infty} g(n)z^n, \tag{1}$$

radius of convergence equal to ∞. It is assumed that the function $g(w)$, where $w=x+iy$, satisfies the following two conditions:

(a) it is single-valued and analytic in the finite w-plane;

(b) it is such that for all values of x and y, one may write

$$| g(x+iy) | < Ke^{\pi|y|}, \tag{2}$$

where K is a positive constant and k is a positive integer. Under these conditions, according to the theorem, $f(z)$ may be expressed in the form

$$f(z) = \int_{-l-1/2}^{\infty} g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} dx - \sum_{m=0}^{l} \frac{g(-m)}{z^m} + \xi(z, l), \tag{3}$$

where l is any positive integer, where the symbol $[\pm z]^x$ means z^x or $(-z)^x$ according as k is odd or even, respectively, and where if $|\arg [\pm z]| < \pi$, we have $\lim_{|z|\to\infty} z^l \xi(z, l) = 0$ for every value of l.

In the present paper we shall consider the situation when conditions (a) and (b) are made somewhat less restrictive. The theorem which we wish to prove is as follows:

THEOREM. *Let the coefficient $g(n)$ in* (1) *satisfy condition* (a) *except for a singularity at the point $w=w_1$, which is not a negative integer; and let inequality* (2) *be satisfied for all values of $|x|$ and $|y|$ sufficiently large. Then* (3) *continues to hold provided one subtracts from the right member the loop integral*

$$\frac{1}{2i}\int_C \frac{g(w)z^w}{e^{k\pi i w}\sin \pi w} dw \tag{4}$$

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[1] *On the character of certain entire functions in distant portions of the plane*, Amer. J. Math. vol. 60 (1938) pp. 561–572.

where the loop C surrounds the point w_1, *and extends to infinity in any convenient direction lying in either the third or the fourth quadrant.*

PROOF. In a concluding remark in his paper, Newsom infers that, under the conditions which we have postulated, the theorem continues to hold provided one subtracts from the right member of (3) a suitable loop integral, which upon examination of the analysis, is seen to be

$$(5)\qquad I = \frac{k!\pi^k}{2i}\int_C \frac{P(w, z)}{(\sin \pi w)^{k+1}}\, dw,$$

where[2]

$$P(w, z) = \sum_{j=1}^{k} b_j e^{r_j \pi i w}\int_0^w e^{-r_j \pi i w} g(w)[\pm z]^w dw,$$

$$r = k + 1 - 2j, \qquad 1 \leqq j \leqq k,$$

$$b_j = \frac{(-1)^{j-1}(2\pi i)^{1-k}}{(k-1)!} C_{j-1}^{k-1}, \qquad C_{j-1}^{k-1} = 1 \text{ when } j = 1.$$

We shall show that the integral I in (5) is equivalent to the integral (4). Evidently we may write

$$P(w, z) = \int_0^w \left\{ \sum_{j=1}^{k} b_j e^{r_j \pi i (w-t)} \right\} g(t)[\pm z]^t dt.$$

However, the sum appearing in the above integrand is readily shown to be equal to

$$\frac{1}{\pi^{k-1}(k-1)!}[\sin \pi(w - t)]^{k-1},$$

and this last expression can be written in the form

$$\frac{1}{\pi^{k-1}(k-1)!}\sum_{j=1}^{k}(-1)^{j-1} C_{j-1}^{k-1}(\sin \pi w)^{k-j}(\cos \pi w)^{j-1}(\cos \pi t)^{k-j}(\sin \pi t)^{j-1}.$$

Introducing into (5) the resulting new form for $P(w, z)$, and making simple reductions, we obtain the equation

$$(6)\qquad I = \frac{k\pi}{2i}\int_C \left\{ \sum_{j=1}^{k} C_{j-1}^{k-1}(-\cot \pi w)^{j-1}\csc^2 \pi w\, \phi_j(w, z) \right\} dw,$$

where

[2] The form of b_j as given here may be seen to be equivalent to that given by Newsom.

$$\phi_j(w, z) = \int_0^w (\cos \pi t)^{k-j}(\sin \pi t)^{j-1} g(t)[\pm z]^t dt.$$

We may simplify (6) by an integration by parts. Consider first the general term of the sum appearing in the integrand. If we let $u = \phi_j(w, z)$, $dv = \pi(-\cot \pi w)^{j-1} \csc^2 \pi w\, dw$, then, taking account of the constant factors, we obtain as an integral of the general term the expression

$$\frac{k}{2ij} C_{j-1}^{k-1} \left\{ (-\cot \pi w)^j \phi_j(w, z) - \int (-\cot \pi w)^j (\cos \pi w)^{k-j} (\sin \pi w)^{j-1} g(w)[\pm z]^w dw \right\}.$$

Moreover the expression $\phi_j(w, z)$ and the indefinite integral here appearing are both such that they can be evaluated along the loop C. Simplifying the above form and summing with respect to j, we have, as an integral of the entire integrand in (6), the sum

$$\frac{k}{2i} \sum_{j=1}^{k} \frac{(-1)^j}{j} C_{j-1}^{k-1} \left\{ (\cot \pi w)^j \phi_j(w, z) - \int (\cos \pi w)^k \frac{g(w)[\pm z]^w}{\sin \pi w} dw \right\}. \tag{7}$$

If we now agree to let C extend to infinity in a direction of either of the third or fourth quadrants, then at the infinite extremities, the function $\cot \pi w$ has the value i. Consequently, upon evaluating (7) along C and combining properly the definite integrals which arise, we arrive at the result

$$I = \frac{k}{2i} \int_C \left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} C_{j-1}^{k-1} (a^j b^{k-j} - b^k) \right\} \frac{g(w)[\pm z]^w}{\sin \pi w} dw, \tag{8}$$

where a and b denote $i \sin \pi w$ and $\cos \pi w$, respectively. The sum appearing in the integrand of (8) is easily shown to be equal to $k^{-1}(b-a)^k$, or $k^{-1}e^{-k\pi i w}$. Substituting this quantity for the sum, we have the integral (4).

The above theorem assumes only one singular point of $g(w)$, but the extension to the case when a finite number of singularities occurs is obvious. Moreover, if a singularity is polar in character, then the corresponding loop integral reduces to the residue of the function

$$\frac{1}{2i} \frac{g(w)[\pm z]^w}{e^{k\pi i w} \sin \pi w}$$

at the pole. Furthermore, if a singularity is a negative integer, say $-n$, then the corresponding term in z^{-n} in (3) is to be suppressed and in its place is to be substituted the proper loop integral.

2. **Asymptotic developments.** The theorem of Newsom, together with the extension just established, finds application in the determination of asymptotic developments for large values of $|z|$ of such integral functions as are defined by (1). It is evident that in order to determine completely the asymptotic development of a given function $f(z)$, one must first find such developments for the integral appearing in the right member of (3), and each loop integral (4) that may occur. The first of these questions is a problem about which we shall not concern ourselves here, but we shall discuss briefly a method by which the second question can often be answered.

If the singularity w_1, of $g(w)$, is algebraic in character, then $g(w)$ can be written in the form $g(w)=(w-w_1)^{\theta}\Phi(w)$, where $\Phi(w)$ is analytic at $w=w_1$, $\phi(w_1)\neq 0$, and where θ is a real constant not equal to zero or a positive integer. One of the transformations $w'=\pm(w-w_1)$ will then transform (4) into an integral of the form

$$I(z,\beta)=\frac{i}{2\pi}\int_{C'}F(w)(-w)^{\beta-1}(-[\pm z])^{-w}dw \tag{9}$$

where β is a constant, and where C' is a loop about the origin $w=0$ (the primes having been dropped), and extending to infinity in a direction lying in either the first or the fourth quadrant. Moreover, $F(w)$ is analytic at the origin, and possesses a convergent series development of the form

$$F(w)=\sum_{n=0}^{\infty}c_n(-w)^n,$$

radius of convergence greater than 0. According to a theorem due to Barnes,[3] if $F(w)$ is bounded in the distant right half w-plane, and if the expressions $(-w)^{\beta-1}$ and $(-[\pm z])^{-w}$ are made precise by suitable

[3] For a full statement and proof of the theorem of Barnes, see Ford, *Asymptotic developments of functions defined by Maclaurin series*, Michigan Science Series vol. 11 (1936) p. 16. This book also contains a proof of the theorem of Newsom for the special case in which $k=1$, together with the extension and numerous applications. For proof of the theorem, see chap. 4; for applications, see chaps. 6 and 7. For further applications of the theorems of Ford and Newsom, and of the theorem in the present paper, see C. G. Fry and H. K. Hughes, *Asymptotic developments of certain integral functions*, Duke Math. J. vol. 9 (1942) pp. 791–802.

definitions, then the loop integral $I(z, \beta)$ in (10) is developable asymptotically in the form

$$I(z, \beta) \sim \sum_{n=0}^{\infty} \frac{c_n}{[\log(-[\pm z])]^{\beta+n}\Gamma(1-\beta-n)}.$$

It thus appears that the presence of an algebraic singularity of $g(w)$ presents no serious difficulty.

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ON SOME FORMULAS INVOLVING THE DIVISOR FUNCTION

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Viggo Brun[1] has proved the formulas

$$T_1(n) - T_2(n) + T_3(n) - \cdots = -\mu(n), \qquad n > 1, \tag{1}$$

$$h(n) = T_1(n) - (1/2)T_2(n) + (1/3)T_3(n) - \cdots$$
$$= \begin{cases} 0 \text{ if } n \text{ is not a prime power,} \\ 1/t \text{ if } n = p^t,\ p \text{ a prime;} \end{cases} \tag{2}$$

where $T_l(n)$ is the number of ways that n can be expressed as a product of l factors, each greater than 1. He obtains them as special cases of combinatorial theorems. Pavel Kuhn[2] has also given proofs but it seems that no one has attempted to give elementary number theory proofs of these formulas. It is the purpose of this note to give such proofs and to point out a few other formulas similar to (1) and (2).

All the formulas which we shall prove can be proved very concisely by using the generating function

$$\sum_{n=1}^{\infty} T_l(n)n^{-s} = \{\zeta(s) - 1\}^l,$$

and some simple properties of the zeta-function.[3] Our number theory

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[1] Netto, *Lehrbuch der Combinatorik*, 2d edition, 1927, chap. 14, especially pp. 276–277.

[2] *Det Kongelige Norske Videnskabers Selskab*, Forhandlinger, 1939.

[3] Interchanging the order of summation we have $\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{l-1}T_l(n)n^{-s} = \sum_{l=1}^{\infty}(-1)^{l-1}\{\zeta(s)-1\}^l = -\zeta(s)^{-1} = -\sum_{n=1}^{\infty}\mu(n)n^{-s}$, and (1) is obtained by comparing coefficients of n^{-s} in the two members. Similarly, (2) follows from $\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{l-1}l^{-1}T_l(n)n^{-s} = \sum_{l=1}^{\infty}(-1)^{l-1}l^{-1}\{\zeta(s)-1\}^l = \log\zeta(s) = \sum_p \log(1-p^{-s})^{-1} = \sum_p\sum_{t=1}^{\infty}t^{-1}p^{-ts}$.