

As for the isomorphism, we need only consider the case where V is defined from G . Obviously, the correspondence $ab^{-1} \leftrightarrow ab$ is one-one from G onto V . Moreover, the correspondent of $ab+cd$ is ah^{-1} where $dc^{-1} = hb^{-1}$ or $h^{-1} = b^{-1}cd^{-1}$. This establishes the isomorphism.

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AN INVARIANT OF INTERSECTION OF TWO SURFACES

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1. Introduction. Projective invariants of several pairs of surfaces have been deduced and characterized geometrically by various authors.¹ In this paper we shall supplement their investigations by studying in ordinary space two surfaces intersecting at an ordinary point with distinct tangent planes.

In §2 we show by analysis the existence of a projective invariant determined by the neighborhood of the second order of the two surfaces at the point of intersection.

The final two sections are devoted to the presentation of projectively, as well as metrically, geometric characterizations of this invariant.

2. Derivation. Suppose that two surfaces S_1, S_2 in ordinary space intersect at an ordinary point O with distinct tangent planes τ_1, τ_2 , and let the common tangent t be distinct from the asymptotic tangents. Let t_1, t_2 be the harmonic conjugate lines of t with respect to the asymptotic tangents of the surfaces S_1, S_2 , respectively, at the point O . If we choose the point O to be the origin, the lines t, t_2, t_1 to be, respectively, the axes x, y, z of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form

$$(1) \quad S_1: y = l_1x^2 + m_1z^2 + \dots,$$

$$(2) \quad S_2: z = l_2x^2 + m_2y^2 + \dots$$

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¹ See the bibliography at the end of the paper.

Let us now consider the most general projective transformation of coordinates which leaves the lines t , t_1 , t_2 unchanged:

$$(3) \quad \begin{cases} x = a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ y = a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ z = a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \end{cases}$$

where a_{ik} are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$(4) \quad \begin{cases} a_{33}l_1^* = a_{22}l_1, & a_{33}m_1^* = a_{44}m_1, \\ a_{44}l_2^* = a_{22}l_2, & a_{44}m_2^* = a_{33}m_2, \end{cases}$$

from which it follows immediately that

$$(5) \quad I = l_1^3 m_2 / l_2^3 m_1$$

is an invariant determined by the neighborhoods of the second order of the two surfaces S_1 , S_2 at the point O .

3. A projectively geometric characterization. Let us consider two pairs of planes, one of which is determined by the line t_2 and the asymptotic tangents of the surface S_1 at the point O , and the other by the line t_1 and the asymptotic tangents of the surface S_2 at the point O . The lines of intersection of these four planes lie on two planes passing through the line t :

$$(6) \quad (l_1 m_2)^{1/2} y \pm (l_2 m_1)^{1/2} z = 0.$$

On the other hand, we observe that the plane sections of the surfaces S_1 , S_2 made by any plane through the line t have, in general, an ordinary contact at the point O . In particular, we can uniquely determine a plane through the line t such that the plane sections have contact of the third order at the point O . The equation of this plane is easily found to be

$$(7) \quad l_2 y - l_1 z = 0.$$

From equations (6), (7) it follows at once that the double ratio of the four planes τ_1 , τ_2 , (7), (6) is equal to

$$(8) \quad (0, \infty, -l_1/l_2, \pm (l_2 m_1/l_1 m_2)^{1/2}) = \pm I^{1/2}.$$

Hence we obtain the following characterization of the invariant I :

Suppose that two surfaces S_1 , S_2 in ordinary space intersect at an ordinary point O with distinct tangent planes τ_1 , τ_2 , and let the common

tangent t be distinct from the asymptotic tangents. Let t_1, t_2 be the harmonic conjugate lines of t with respect to the asymptotic tangents of the surfaces S_1, S_2 , respectively, at the point O ; and let π_1 be one of the two planes through t on which lie the lines of intersection of two pairs of planes, one of which is determined by t_2 and the asymptotic tangents of the surface S_1 at the point O , and the other by t_1 and the asymptotic tangents of the surface S_2 at the point O . If π_2 be the unique plane through the line t such that the plane sections of the surfaces S_1, S_2 made by π_2 have contact of the third order at the point O , then the projective invariant I associated with the surfaces S_1, S_2 at the point O is equal to the square of the double ratio $(\tau_1 \tau_2, \pi_2 \pi_1)$.

4. A metrically geometric characterization. It is deemed worthwhile to find a simple metrical characterization of the projective invariant I . For this purpose let us make a projective transformation which leaves the point O unchanged and carries the lines t, t_1, t_2 into three mutually perpendicular lines $\bar{l}, \bar{l}_1, \bar{l}_2$. Let \bar{S}_1, \bar{S}_2 be the transformed surfaces of S_1, S_2 , and $\bar{x}, \bar{y}, \bar{z}$ the nonhomogeneous Cartesian coordinates of a point in space referred to the orthogonal coordinate system determined by the lines $\bar{l}, \bar{l}_1, \bar{l}_2$; then the power series expansions of the surfaces \bar{S}_1, \bar{S}_2 in the neighborhood of the point O may be written in the form

$$(9) \quad \bar{S}_1: \bar{y} = \bar{l}_1 \bar{x}^2 + \bar{m}_1 \bar{z}^2 + \dots,$$

$$(10) \quad \bar{S}_2: \bar{z} = \bar{l}_2 \bar{x}^2 + \bar{m}_2 \bar{y}^2 + \dots,$$

and the projective invariant I takes the form

$$(11) \quad I = \bar{l}_1^3 \bar{m}_2 / \bar{l}_2^3 \bar{m}_1.$$

Let \bar{K}_1, \bar{K}_2 be the total curvatures of the surfaces \bar{S}_1, \bar{S}_2 at the point O ; and \bar{R}_1, \bar{R}_2 the radii of curvature at the point O of the curves in which the tangent planes $\bar{z}=0, \bar{y}=0$ intersect with the surfaces \bar{S}_1, \bar{S}_2 respectively. Then it can be demonstrated that

$$(12) \quad \begin{cases} \bar{K}_1 = 4\bar{l}_1 \bar{m}_1, & \bar{K}_2 = 4\bar{l}_2 \bar{m}_2, \\ \bar{R}_1 = 1/2\bar{l}_1, & \bar{R}_2 = 1/2\bar{l}_2, \end{cases}$$

and therefore that

$$(13) \quad I = \bar{R}_2^4 \bar{K}_2 / \bar{R}_1^4 \bar{K}_1.$$

Thus we obtain the following metrical characterization of the projective invariant I .

Let K_1, K_2 be the total curvatures of the surfaces S_1, S_2 at the point O ,

and R_1, R_2 the radii of curvature at O of the curves in which the tangent planes τ_2, τ_1 intersect the surfaces S_1, S_2 , respectively. Then $R_2^4 K_2 / R_1^4 K_1$ is the projective invariant I of the surfaces S_1, S_2 above indicated.

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