OSKAR BOLZA—IN MEMORIAM

Oskar Bolza came to the United States from Germany in 1888, and returned in 1910 to make his home in Freiburg im Baden during the later years of his life. He was an influential member of our American mathematical community for twenty-two years, including those significant ones during which the American Mathematical Society was founded and had its early growth. He was one of the founders of the Chicago Section of the Society, was a member of the Council of the Society in 1900–1902 and vice president in 1904. He was beloved by his students and associates. He was born on May 12, 1857 in Bergzabern, Germany, and word reached this country in the autumn of 1942 through the American Red Cross that he died peacefully on July 5, 1942, presumably in Freiburg. The following pages are devoted to a sketch of his life and scientific activities. They have unusual significance for our American mathematical community.

Bolza's father was in the judicial service in Germany and the family dwelt in various places in southern Germany during Bolza's childhood years. In 1873 they made Freiburg their permanent hôme, and the ties which bound Bolza to that city were thereafter always very strong. His mother was a daughter of Friedrich Koenig, the inventor of a rapid printing press in the early part of the nineteenth century. The manufacture of printing presses proved to be profitable through the years, and so far as is known Bolza's life was free from financial worries.

Even in his early years Bolza was an industrious and independent student. His first strong interests seemed to be the beginning of a career in languages and comparative philology. But in 1873 he attended an academy in Neuchatel and later the Gymnasium in Freiburg, in both of which he had excellent instructors in mathematics. There for the first time he became enamoured of the subject, and mathematics remained his principal interest during all but a few years of the rest of his life.

In the autumn of 1875 Bolza went to Berlin to begin his university student career. At that time it was expected that he would ultimately succeed to a share in the management of the family printing press factory. But his father wanted him to have a liberal as well as a technical education, and for that reason he first attended both the University and the Gewerbeakademie in Berlin. It soon became clear that this double program was impractical, and that Bolza's interests were likely to lie much more in pure science than in business. He at first turned to physics as a possible principal life activity, and during the years 1876–1878 he studied in that field under Kirchoff and Helmholz in Berlin, Quinke in Heidelberg, and Kundt in Strassburg. He even undertook to write a doctor's dissertation in physics under Kundt. But experimental work had always been burdensome to him and in the autumn of 1878 he decided finally, and certainly fortunately in view of his later attainments, to make pure mathematics his specialty.

The next three years Bolza pursued his interest in pure mathematics under Christoffel and Reye in Strassburg, Weierstrass in Berlin, Schwarz in Göttingen, and Weierstrass again. He was not pressed financially and had the approval of his father in his effort to prepare himself for a university career. But in the autumn of 1880 he found himself still far from his doctor's degree and already in his eleventh university semester. He determined therefore to prepare himself for the Staatsexamen which in Germany was a prerequisite for Gymnasium teaching though not for a university career. It was three years before his examination was passed and his year of practice teaching in the Gymnasium at Freiburg completed. He did well in this undertaking, and was successful and greatly interested in his teaching work.

For two years, 1883–1885, he worked privately on his doctor's dissertation in Freiburg. The subject which he had chosen was the determination of hyperelliptic integrals which can be reduced to elliptic integrals by a transformation of the third degree. He found the solution of his problem in 1885 just as a more elegant one by Goursat appeared in the Comptes Rendus. Nothing daunted Bolza attempted and found it easy to solve the corresponding problem for transformations of the fourth degree. His solution was published promptly in the Berichten der Freiburger Naturforschenden Gesellschaft. It was his first published paper.

In 1885 and 1886 Bolza studied again at Berlin under Fuchs and Kronecker. He extended his results on the transformation problem and after some correspondence with Schwarz and Klein in Göttingen his paper was accepted by the latter as a dissertation for the doctor's degree. Bolza's results were undoubtedly of great interest to Klein since they were closely related to some which Klein had himself recently published. In June 1886 Bolza passed his examination and was awarded the doctor's degree at the age of 29.

The next year, 1886–1887, Bolza and his intimate student friend Heinrich Maschke had a kind of private seminar with Klein in Göttingen. The three met once a week at Klein's house and planned and reported on research projects of common interest. It was a great experience for the younger men, but it had an unexpected effect upon

Bolza. Klein had an altogether exceptional genius and breadth of view in mathematics, and his personality was justifiably confident and overpowering. Bolza was a modest person, and as a result of his association with Klein he experienced a loss of confidence in himself which was completely unjustified, as later years have well shown.

It was at this stage in his career that Bolza debated seriously with himself what his future might turn out to be. As a student in Berlin he had made two exceptionally intimate young friends, the mathematician Heinrich Maschke mentioned above and the physicist Franz Schulze-Berge. Both of them had taken positions as gymnasium instructors but had found the teaching exceedingly heavy with little opportunity for scientific work. In the spring of 1887 Schulze-Berge decided to try his fortune in the United States, and upon his arrival here he promptly found a place in the experimental laboratory of Thomas Edison in New Jersey. This experience and the persuasiveness of two American mathematical students in Göttingen, M. W. Haskell and F. N. Cole, decided Bolza to follow the example of his friend, and in the spring of 1888 he landed in Hoboken.

Not long after his arrival he was greatly encouraged to find a place as "Reader in Mathematics" at Johns Hopkins University beginning in January 1889. Before the college year was out he was appointed "Associate in Mathematics" in the newly founded Clark University of Worcester, Massachusetts. The appointment began in October 1889 when the University first opened. Clark University had been founded as a graduate school with funds which soon proved to be insufficient. The faculty became restive and dissatisfied, and in 1892 when the new University of Chicago was opened, President Harper was able to persuade a number of them to move to his still newer institution. Bolza was one of these. He was made an associate professor, beginning January 1, 1893, with the understanding that he should be advanced to a full professorship after one year. He had also persuaded President Harper and E. H. Moore, the youthful head of the department of mathematics, to appoint Maschke to an associate professorship, Maschke having meanwhile also come to the United States.

Bolza was highly successful in his teaching and research during the eighteen years which he spent at the University of Chicago. The trio, Moore, Bolza, Maschke, had a profoundly stimulating influence on each other and on the continually numerous group of graduate students who went to Chicago to listen to them, a group now widely distributed among the faculties of our American universities. Among notable mathematical events which took place during this period were the International Mathematical Congress at the World's Fair in Chicago in 1893 and the immediately following inspiring Colloquium Lectures by Klein at Northwestern University, the transformation of the New York Mathematical Society into the American Mathematical Society, the foundation of the Chicago Section and other sections of the Society, and the foundation of the Transactions of the American Mathematical Society in 1900. In all of these activities Moore, Bolza, and Maschke took a prominent part. At the third colloquium of the Society in Ithaca in August 1901, Bolza lectured in beautiful fashion on the calculus of variations, a subject then much in vogue because of the then recent appearance of Kneser's book in that field.

In March of 1908 Maschke died and one of the strongest ties which bound Bolza to Chicago was thereby broken. During his sojourn in Chicago Bolza had frequently refreshed his relations with his relatives and friends in Germany by journeys to Freiburg during his vacation periods. Furthermore he had the feeling that he should make way for some of the large group of young and able American mathematicians then becoming established in increasing numbers in American mathematical departments. Soon after Maschke's death, therefore, Bolza decided to return to Freiburg to spend his later years in his original home environment. The mathematical department at the University of Chicago released him with great reluctance and with the title Nonresident Professor of Mathematics which he held until his death.

In Freiburg Bolza was appointed Ordentlicher Honorar professor at the University there, at the instigation of his friend, Lüroth, and for several years he continued his mathematical lectures and research with great success. In the summer quarter of 1913 he lectured again at the University of Chicago to the great satisfaction of his friends and hearers. But the war of 1914–1918 had a disastrous effect upon Bolza's mathematical research. He continued his lectures until 1926 but from as early as 1917, under the influence of several very good friends, his interest turned strongly to religious psychology and to Sanskrit which he studied for the purpose of reading at first hand literature concerning the religious systems of India. The result of these activities was a book entitled *Glaubenlose Religion* published in 1930 under the pseudonym F. H. Marneck.

After an interruption of three years Bolza resumed his mathematical lectures in 1929 at the University of Freiburg before the largest classes he had ever had. In one case ninety students listened to his lectures on differential equations. But the flood of students ebbed away there as elsewhere during succeeding years, and in 1933 at the age of 76 Bolza finally gave up lecturing and turned again to his psychological studies. Then a very interesting thing happened to him. Professor J. H. MacDonald, one of his Ph.D. students in 1901, sojourned for two weeks in Freiburg, and the two of them took long walks together during which they discussed again the subject which had long before engaged their interest, the transformation of hyperelliptic into elliptic integrals by transformations of specified degrees. Bolza was inspired to attempt anew some research in this field of his earliest mathematical activity and he succeeded in completing the solution of a problem which had long interested him much. His results were published in the Mathematische Annalen in 1935 in the paper numbered 57 in the bibliography below. In 1938 Bolza published one further study in religious psychology (no. 62 in the bibliography). It was his last published paper.

We know, however, that Bolza's enthusiasm for mathematics continued to the end. In a letter to Professor Arnold Dresden, written in 1939 at the age of 82, Bolza tells of his interest in studying the foundations of geometry with his friend and colleague in Freiburg, the geometer Heffter. According to Bolza's own statement his intention was to try to fill a yawning gap in his earlier mathematical training.

Since 1939 we have had little news from Bolza until the announcement in the autumn of 1942 of his death. But the impressions of his able and kindly personality which he has left behind, and the records of his inspiring lectures and research, are a part of our American mathematical traditions which will be with us always.

In the pages to follow in this biography some indications will be given of the scope and character of Bolza's scientific work.

The Scientific Work of Oskar Bolza

The scope and emphasis of Bolza's research are indicated roughly by the following table which lists for each of his principal fields the papers which he published and the years during which he was active in that field. The numbers refer to the bibliography at the end of this paper.

I. Elliptic and hyperelliptic integrals and functions: 1–6, 13–15, 17–20, 22, 23, 25, 60; 1885–1901, 1936.

II. Miscellaneous: 7–12, 16, 21, 34, 40, 41, 58, 61, 62; 1888–1938.

III. Calculus of variations: 24, 26–33, 35–39, 42–53, 56, 57, 59; 1901–1933.

IV. Integral equations and general analysis: 46, 54, 55; 1910–1914.

In Group I Bolza studied first the problem of determining hyperelliptic integrals which are reducible to elliptic integrals by transformations especially of the third and fourth degrees. It was his first research interest, the field of his dissertation sponsored by Klein in Göttingen, a problem often studied by leading mathematicians in the latter part of last century. Beginning with his sojourn under Klein in Göttingen in 1886, and continuing during something more than a decade following his move to the United States, Bolza published eleven papers concerning hyperelliptic Θ-, σ-, and $\wp$-functions and related subjects. In these he emphasized the elliptic theory and reformulated it in numerous instances to show how it appears as a special case of the hyperelliptic theory, and he carried over to the more general case many theorems and formulas well established in connection with the theory of elliptic functions. He was an able follower of his great teachers, Weierstrass and Klein, in this field.

When Bolza first went to Johns Hopkins in 1889 he was asked by the noted astronomer, Simon Newcomb, then head of the department of mathematics, to lecture on the theory of substitution groups and its applications to algebraic equations. In 1891 the lectures were published by invitation in the paper numbered 8 below, and in neighboring years Bolza wrote three further papers on the subject. One was a review of Cole's translation of Netto's *Theory of substitutions*, a book which aroused much interest in this country. The review was most thorough and scholarly, quite characteristic of Bolza, and much more comprehensive and critical than most reports of this sort.

Under the title "Miscellaneous" above I have grouped two papers on linear differential equations of the second order, Bolza's two publications concerning religious psychology, and other papers. But I wish to mention especially his autobiography *Aus meinem Leben* published privately in 1936. I had some time before urged him to write an account of his life and interests and was overjoyed when the result reached me. It is the source of much of the data in the sketch of his life presented here, and many readers besides his own students and colleagues will find great encouragement and interest in it.

By far the greatest of Bolza's interests was the calculus of variations, as indicated by the twenty-nine publications under III above. He was a student in the famous course of Weierstrass on the calculus of variations at the University of Berlin in 1879, but did not pursue the subject actively further until 1901 when he was invited to give a series of colloquium lectures at the summer meeting of the American Mathematical Society in Ithaca. He chose the calculus of variations as his subject. It was beautifully presented. The lectures much amplified, published in 1904 in the book numbered 33 in the bibliography below, contained the contributions to that date, especially of Weierstrass, Kneser, and Hilbert, for problems of the calculus of variations in the plane in both non-parametric and parametric form, enriched by the able presentation of Bolza himself. This was only his beginning. He published in 1909 his *Vorlesungen über Variationsrechnung*, numbered 44 below, which is now a classic, indispensable to every scholar in the field, and much wider in its scope than his earlier book. The later volume contains besides the theory of problems in the plane, the theory of the problem of Lagrange with fixed end points, and an introduction to the problem of the calculus of variations for double integrals.

One of Bolza's important contributions was his extension and application of existence theorems for implicit functions and for solutions of differential equations. Before his time most theorems concerning implicit functions defined by equations of the form

$$f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) = 0 \qquad (i = 1, \cdots, n)$$

were based upon the hypothesis that there is a single solution $(x; y) = (a; b)$ of these equations, and the purpose of the theorems was to show that for x in a neighborhood of a there exists a uniquely defined system of solutions

$$y_i(x_1, \cdots, x_m) \qquad (i = 1, \cdots, n)$$

near the point $y = b$. Bolza assumed a set P of solutions $(x; y) = (a; b)$, instead of a single one, and proved the existence of solutions defined over a neighborhood of the projection X of P in x-space. Similarly for a system of differential equations

$$f_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (i = 1, \cdots, n)$$

he assumed the existence of a solution arc $y_i(x)$ $(x_1 \leqq x \leqq x_2;$ $i = 1, \cdots, n)$ and proved that it could be imbedded in an n-parameter family of neighboring solutions. These theorems or their equivalents, as Bolza showed and as other students of the subject now well recognize, are essential in many instances to a rigorous theory of the calculus of variations.

Two of Bolza's papers, 53 and 56 in the bibliography below, have interested me especially. In them he presents a formulation of a new and very general problem of the calculus of variations which includes as special cases most of the problems which have been studied hitherto, and which has come to be known as the problem of Bolza.

The problem in parametric form is to find in a class of arcs $y_i(t)$ $(i=1, \cdots, n; t_0 \leqq t \leqq t_1)$ satisfying equations of the form

$$\phi_\alpha(y, y') = 0 \qquad (\alpha = 1, \cdots, p)$$

$$\psi_\beta(y) = 0 \qquad (\beta = 1, \cdots, q)$$

and end conditions of the form

$$\chi_\gamma[y(t_0), y(t_1)] = 0 \qquad (\gamma = 1, \cdots, r)$$

one which minimizes a sum

$$I = G[y(t_0), y(t_1)] + \int_{t_0}^{t_1} f(y, y')dt.$$

In these expressions the symbol y stands for the set $(y_1, \cdots, y_n)$, and similarly for y'. Bolza deduced first necessary conditions for a minimum and showed how the problems of Lagrange and Mayer are unified under his theory. He distinguishes with great clarity between so-called normal and abnormal minimizing arcs in cases much more general than had been considered before. In the second of the two papers he studies problems for which some of the side conditions written above are inequalities in place of equalities. It is impossible to describe or even mention here all of the contributions which Bolza made to the calculus of variations. He added liberally to the theory in the plane and to the problem of Lagrange, and improved them in many important respects. His methods are models which have greatly influenced many workers in the field.

The group of papers listed as IV above is a small one. Bolza was a great friend and admirer of E. H. Moore. The paper 46 is devoted to the formulation and proof of a theorem in *General analysis* which generalizes and unifies a multiplier rule of the calculus of variations and a theorem of elementary analytic geometry in the spirit so often formulated by Moore. In the summer of 1913 Bolza gave one of his inimitable lecture courses at the University of Chicago on integral equations. The lectures were written up by W. V. Lovitt who afterward published them with modifications in book form (see no. 54 of the bibliography). When Bolza returned to Germany after his summer in Chicago he had the strong feeling that Moore's *General analysis* and the accompanying theory of integral equations had been stated in print only very concisely, and that they should be much better known. He therefore published in the Jahresbericht an introduction of some fifty-six pages to Moore's theory, the paper numbered 55 below. It is written in Bolza's finest style and should be most helpful and suggestive to all students of the theory.

It would require a book to describe all of the research which Bolza published, even in the calculus of variations alone, and I shall content myself with the brief indications in the paragraphs which precede. His work was always meticulously precise, and was usually accompanied by a most scholarly and appreciative analysis of publications in the same neighborhood which preceded his own. Wherever his interest was aroused he has left valuable and indelible impressions in the field of mathematical research.

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