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A GENESIS FOR CESÀRO METHODS

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1. **Introduction.** The Cesàro methods C_r, introduced by Cesàro[1] because of their applicability to Cauchy products of series, constitute the most publicized class of methods of summability.

The regular Nörlund methods[2] of summability constitute one of the two most publicized general classes of consistent methods of summability. The regular Hurwitz-Silverman-Hausdorff methods constitute the other.

This note proves the following theorem.

THEOREM. *The Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hurwitz-Silverman-Hausdorff methods.*

Thus if the Cesàro methods had not been previously introduced into mathematical literature, they could be defined and exploited as the unique class of methods of summability enjoying all of the properties of Nörlund methods and all of the properties of Hurwitz-Silverman-Hausdorff methods.

In §4, it is shown that the only methods which are both Riesz methods and Hurwitz-Silverman-Hausdorff methods are methods Γ_r closely related to the methods C_r.

2. **Nörlund methods.** Each sequence $p_0, p_1, \cdots$ of real or complex constants for which $P_n \equiv p_0+p_1+ \cdots +p_n \neq 0$ for each n defines a Nörlund method of summability by means of which a sequence $s_0, s_1, \cdots$ is summable to σ if $\sigma_n \rightarrow \sigma$ where

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[1] E. Cesàro, *Sur la multiplication des series*, Bull. Sci. Math. (2) vol. 14 (1890) pp. 114–120.

[2] N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Arsskrift (2) vol. 16 (1919).

$$(1)\qquad \sigma_n = (p_n s_0 + p_{n-1}s_1 + \cdots + p_1 s_{n-1} + p_0 s_n)/P_n.$$

The class of Nörlund transformations (1) is identical with the class of triangular matrix transformations

$$(2)\qquad \sigma_n = \sum_{k=0}^{n} a_{nk}s_k$$

for which

$$(3)\qquad a_{nn} \neq 0, \qquad n = 0, 1, 2, \cdots,$$

$$(4)\qquad \sum_{k=0}^{n} a_{nk} = 1, \qquad n = 0, 1, 2, \cdots,$$

and, for each $q=0, 1, 2, \cdots$, there is a constant b_q such that

$$(5)\qquad a_{n,n-q} = b_q a_{nn}, \qquad n \geqq q.$$

3. **Hurwitz-Silverman-Hausdorff methods.** These methods (hereafter the HSH methods) constitute the class of triangular matrix transformations which commute with the arithmetic mean transformation

$$(6)\qquad \sigma_n = (s_0 + s_1 + \cdots + s_n)/(n+1)$$

and hence also with each other.

As Hurwitz and Silverman[3] and Hausdorff[4] have shown, with each method HSH there is associated a generating sequence $\lambda_0, \lambda_1, \cdots$ such that the transformation takes the form

$$(7)\qquad \sigma_n = \sum_{j=0}^{n} (-1)^j C_{n,j}\lambda_j \sum_{k=0}^{j} (-1)^k C_{j,k}s_k$$

or

$$(8)\qquad \sigma_n = \sum_{k=0}^{n}\left[C_{n,k}\sum_{j=0}^{n-k}(-1)^j C_{n-k,j}\lambda_{j+k}\right]s_k.$$

Assume that (8) is a Nörlund transformation, and let the quantity in brackets in (8) be denoted by a_{nk}. Then, $a_{nn}=\lambda_n$ so, by (3) and (4), $\lambda_n \neq 0$ for each n and $\lambda_0 = 1$. Moreover

$$(9)\qquad a_{n,n-1} = n(\lambda_{n-1} - \lambda_n)$$

[3] W. A. Hurwitz and L. L. Silverman, *On the consistency and equivalence of certain definitions of summability*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 1–20.

[4] F. Hausdorff, *Summationsmethoden und Momentfolgen.* I and II, Math. Zeit. vol. 9 (1921) pp. 74–109 and 280–299.

and hence (5) with $q=1$ guarantees existence of a constant (which we call r instead of q_1) such that

$$n(\lambda_{n-1} - \lambda_n) = r\lambda_n, \qquad n \geqq 1. \tag{10}$$

Here r cannot be a negative integer; otherwise one could set $n=-r$ in (10) and contradict $\lambda_{-r-1} \neq 0$. Therefore

$$\lambda_n = (n/(n+r))\lambda_{n-1}, \qquad n = 1, 2, \cdots. \tag{11}$$

Since $\lambda_0 = 1$, (11) implies that

$$\lambda_1 = 1/(1+r), \quad \lambda_2 = (2/(2+r))\lambda_1 = 1\cdot 2/(1+r)(2+r) \tag{12}$$

and in general

$$\lambda_n = 1\cdot 2\cdot 3 \cdots n/(1+r)(2+r)\cdots(n+r) = n!r!/(n+r)!, \qquad n = 0, 1, \cdots, \tag{13}$$

where $r!$ is the factorial function, $\Gamma(r+1)$, defined for all complex r except $-1, -2, -3, \cdots$.

By use of the familiar identity

$$\sum_{k=0}^{n} C_{n-k+r-1,r-1} = C_{n+r,r}, \qquad r \neq -1, -2, \cdots, \tag{14}$$

we obtain the familiar fact that for each $r \neq -1, -2, \cdots$ the Cesàro transformation C_r,

$$\sigma_n = \sum_{k=0}^{n} [C_{n-k+r-1,r-1}/C_{n+r,r}]s_k, \tag{15}$$

is the one and only Nörlund transformation for which $a_{nn} = \lambda_n = n!r!/(n+r)!$. This completes the proof of the fact that the only HSH transformations which are Nörlund transformations are the Cesàro transformations.

It is only when $r=0$ or r is a complex number with a positive real part that C_r is regular and can be written in the Hausdorff form

$$\sigma_n = \int_0^1 \sum_{k=0}^{n} C_{n,k} t^k (1-t)^{n-k} s_k d\chi(t) \tag{16}$$

where

$$\chi(t) = 1 - (1-t)^r. \tag{17}$$

For each complex r not a negative integer, the more general Hurwitz-Silverman transformation (8) becomes C_r when $\lambda_n = n!r!/(n+r)!$.

4. Riesz methods. Each sequence $p_0, p_1, \cdots$ for which $P_n \equiv p_0 + p_1 + \cdots + p_n \neq 0$ for each n determines a Riesz transformation

$$(18) \qquad \sigma_n = (p_0 s_0 + p_1 s_1 + \cdots + p_n s_n)/P_n, \qquad n = 0, 1, 2, \cdots .$$

If one of the two transformations

$$(19) \qquad \sigma_n' = \sum_{k=0}^{n} a_{nk} s_k, \qquad \sigma_n'' = \sum_{k=0}^{n} a_{n,n-k} s_k$$

is a Nörlund transformation, the other is a Riesz transformation. It can be shown, by a suitable modification of our treatment of HSH and Nörlund methods, that if A is both an HSH method and a Riesz method, then there is a constant r not a negative integer such that A has the form

$$(20) \qquad \sigma_n = \sum_{k=0}^{n} [C_{k+r-1,r-1}/C_{n+r,r}] s_k.$$

Thus *the only methods which are simultaneously HSH methods and Riesz methods are the methods* Γ_r *defined by* (20). For each $r \neq -1, -2, \cdots$, the transformation Γ_r is obtained from C_r by reversing the order in which the elements $a_{nk}^{(r)}$ are applied to $s_0, s_1, \cdots$. The HSH sequence λ_n generating Γ_r is $\lambda_0 = a_{00} = 1$ and

$$(21) \qquad \lambda_n = a_{nn} = r/(n + r), \qquad n > 0.$$

The following discussion applies only to *regular* transformations. The elements a_{nk} of the matrix of the Hausdorff transformation generated by $\chi(t)$ are given by

$$(22) \qquad a_{n,k} = \int_0^1 C_{n,k} t^k (1 - t)^{n-k} d\chi(t).$$

If (22) holds, then

$$a_{n,n-k} = \int_0^1 C_{n,k} t^{n-k} (1 - t)^k d\chi(t)$$
$$= - \int_0^1 C_{n,k} u^k (1 - u)^{n-k} d_u \chi(1 - u)$$

and hence

$$(23) \qquad a_{n,n-k} = \int_0^1 C_{n,k} t^k (1 - t)^{n-k} d_t [1 - \chi(1 - t)].$$

It follows that if $\chi(t)$ generates one of the transformations in (19), then $\chi_1(t) \equiv 1 - \chi(1 - t)$ generates the other. Suppose now that $\chi(t)$

generates a regular Riesz method. Then $1-\chi(1-t)$ generates a Nörlund method, which must be a Cesàro method C_r. Hence $1-\chi(1-t)=1-(1-t)^r$ and $\chi(t)=t^r$. Since t^r generates a regular HSH method only when $\Re r>0$, and since t^r generates the regular Riesz method Γ_r when $\Re>0$, we have proved the following result. *The methods Γ_r for which $\Re r>0$ are the only regular methods of summability which are simultaneously Riesz methods and HSH methods.*

The identity

$$\frac{r!n!}{(n+r)!}=\frac{(r-1)!n!}{(n+r-1)!}\frac{r}{n+r}, \qquad r\neq 0,-1,\cdots, \tag{24}$$

implies the identities $C_r=C_{r-1}\Gamma_r=\Gamma_rC_{r-1}$ and $\Gamma_r=C_rC_{r-1}^{-1}=C_{r-1}^{-1}C_r$ involving the methods Γ_r and C_r; this is a consequence of the fact (Hurwitz-Silverman, loc. cit. p. 7) that if λ_n' generates A' and λ_n'' generates A'', then $\lambda_n'\lambda_n''$ generates $A'A''$. From the fact that C_r and the Hölder method H_r are equivalent ($C_r\sim H_r$) when $\Re r>-1$ we obtain, when $\Re r>0$, the familiar formulas

$$C_r\sim H_r=H_{r-1}H_1\sim C_{r-1}H_1=C_1C_{r-1} \tag{25}$$

and $C_rC_{r-1}^{-1}\sim C_1$. This gives the fact, proved by Hausdorff, loc. cit., that Γ_r is equivalent to C_1 when $\Re r>0$.

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