NOTES ON LEGENDRE POLYNOMIALS

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1. Introduction. We shall obtain three results. The first is a generating function for the Legendre polynomials that lies between the classical

$$(1 - 2xW + W^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)W^n$$

and a special case of a well known result of Bateman.[1] The second is an expression for $P_n(\cos \alpha)$ as a series in $P_k(\cos \beta)$ with α and β unrelated. Special cases of the latter are known. The third result is a relation in integral form between the Hermite and Legendre polynomials.

2. A generating function for $P_n(x)$. From Laplace's first integral,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x + (x^2 - 1)^{1/2} \cos \beta]^n d\beta,$$

we find by direct expansion and the use of Wallis' formula the known[2] result

$$P_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(1 - x^2)^k x^{n-2k}}{2^{2k}(k!)^2(n - 2k)!}, \tag{1}$$

where [ ] is the greatest integer symbol.

Examination of (1) in the light of the identity

$$\left(\sum_{n=0}^{\infty} a_n y^{2n}\right)\left(\sum_{n=0}^{\infty} b_n y^n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_k b_{n-2k} y^n, \tag{2}$$

and of the power series for the Bessel function $J_0(y)$,

$$J_0(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{2^{2n}(n!)^2},$$

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[1] H. Bateman, *A generalization of the Legendre polynomial*, Proc. London Math. Soc. (2) vol. 3 (1905) pp. 111–123. I am much indebted to Professor Bateman for pointing out that the generating function in (3) may be found in the work of Catalan, F. H. Jackson, and others. This relation seems to have been less widely used than it deserves. The short, possibly new, derivation in §2 may contribute to an understanding of the material in §§3 and 4.

[2] E. W. Barnes, *On generalized Legendre functions*, Quarterly Journal of Pure and Applied Mathematics vol. 39 (1908) pp. 97–204. See p. 120.

reveals at once that

$$e^{tx}J_0(t(1-x^2)^{1/2}) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \tag{3}$$

or

$$e^{t\cos\theta}J_0(t\sin\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{t^n}{n!}\cdot \tag{4}$$

3. **The polynomials $\phi_n(z)$.** In the study of the contribution to electron scattering of a freely rotating group within molecules comprising a jet of gas, Dr. Jerome Karle encountered[3] the polynomials

$$\phi_n(z) = \frac{1}{\pi}\int_0^{\pi}(1 + z\cos\beta)^n d\beta.$$

With the aid of Laplace's first integral, we may write

$$\phi_n(z) = (1-z^2)^{n/2}P_n\left(\frac{1}{(1-z^2)^{1/2}}\right). \tag{5}$$

It happens that the $\phi_n(z)$ have a very simple generating function. For the modified Bessel function $I_0(y)$ it is known that

$$I_0(y) = \frac{1}{\pi}\int_0^{\pi} e^{y\cos\beta}d\beta.$$

Hence

$$e^tI_0(tz) = \frac{1}{\pi}\int_0^{\pi} e^{t(1+z\cos\beta)}d\beta,$$

or

$$e^tI_0(tz) = \sum_{n=0}^{\infty}\phi_n(z)\frac{t^n}{n!}\cdot \tag{6}$$

This generating function may also be obtained from equations (1), (3), and (5) above.

4. **A formula for $P_n(\cos\alpha)$.** We have shown elsewhere[4] that if

$$e^tG(xt) = \sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!},$$

then

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[3] Up to the present time computational difficulties independent of the $\phi_n(z)$ have prevented application of the results.

[4] *Certain generating functions and associated polynomials*, now in the hands of the editors of the Amer. Math. Monthly.

$$g_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1-y)^{n-k} g_k(x).$$

Therefore, we may write

$$\phi_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1-y)^{n-k} \phi_k(x). \tag{7}$$

From (5) it follows that $\phi_n(i \tan \alpha) = \sec^n \alpha P_n(\cos \alpha)$. Now, let us put $x = i \tan \beta$ and $y = \tan \alpha / \tan \beta$ in (7). It becomes

$$\sec^n \alpha P_n(\cos \alpha) = \sum_{k=0}^{n} C_{n,k} \frac{\tan^k \alpha}{\tan^n \beta} (\tan \beta - \tan \alpha)^{n-k} \sec^k \beta P_k(\cos \beta).$$

The above may be simplified by elementary means to the form

$$P_n(\cos \alpha) = \left(\frac{\sin \alpha}{\sin \beta}\right)^n \sum_{k=0}^{n} C_{n,k} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha}\right]^{n-k} P_k(\cos \beta). \tag{8}$$

5. **Special cases of the above formula.** It is interesting to note certain special cases of (8), either because they yield well known results or because of their intrinsic neatness.

Using $\beta = 2\alpha$, we get

$$(2 \cos \alpha)^n P_n(\cos \alpha) = \sum_{k=0}^{n} C_{n,k} P_k(\cos 2\alpha).$$

Using $\alpha = \pi/2 + \beta$, we get

$$\sin^n \beta P_n(\sin \beta) = \sum_{k=0}^{n} (-1)^k C_{n,k} \cos^k \beta P_k(\cos \beta).$$

Finally, using $\beta = -\alpha$,

$$P_n(\cos \alpha) = \sum_{k=0}^{n} (-1)^k C_{n,k} (2 \cos \alpha)^{n-k} P_k(\cos \alpha),$$

or

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} (2x)^{n-k} P_k(x),$$

which in symbolic notation is $P_n(x) \doteqdot \{2x - P(x)\}^n$.

6. **A formula for the Hermite polynomial.** Either the generating function (6) or the integral definition of the $\phi_n(z)$ may be used to obtain

$$\phi_n(z) = n! \sum_{k=0}^{[n/2]} \frac{z^{2k}}{2^{2k}(k!)^2(n-2k)!}.$$

The formula above naturally reminds one of the Hermite polynomial,

$$H_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}.$$

Indeed, we may write

$$\phi_n(2it^{1/2}) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{t^k}{(k!)^2(n-2k)!},$$

take the Laplace transform of both sides, and get

$$\int_0^\infty e^{-st}\phi_n(2it^{1/2})dt = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{s^{-k-1}}{k!(n-2k)!} = s^{-(n+2)/2}H_n(s^{1/2}/2).$$

Therefore,

$$H_n(x) = (2x)^{n+2} \int_0^\infty e^{-4x^2t}\phi_n(2it^{1/2})dt,$$

or, with $4t=\alpha$,

$$H_n(x) = 2^n x^{n+2} \int_0^\infty e^{-\alpha x^2}\phi_n(i\alpha^{1/2})d\alpha.$$

But, from (5) it follows that

$$\phi_n(i\alpha^{1/2}) = (1+\alpha)^{n/2}P_n\left(\frac{1}{(1+\alpha)^{1/2}}\right).$$

Hence, it is possible to write

$$H_n(x) = 2^n x^{n+2} \int_0^\infty e^{-\alpha x^2}(1+\alpha)^{n/2}P_n\left(\frac{1}{(1+\alpha)^{1/2}}\right)d\alpha,$$

or

$$H_n(x) = 2^{n+1}x^{n+2}e^{x^2} \int_0^1 e^{-x^2/\beta^2}\beta^{-n-3}P_n(\beta)d\beta.$$

This result can also be put in the form

$$H_n(x) = 2^{n+1}e^{x^2} \int_x^\infty e^{-t^2}t^{n+1}P_n\left(\frac{x}{t}\right)dt,$$

resembling that of Curzon[5] but apparently simpler.

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[5] H. E. J. Curzon, *On a connexion between the functions of Hermite and the functions of Legendre*, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 236–259.