A SUFFICIENCY THEOREM FOR DIFFERENTIAL SYSTEMS

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1. Introduction. This paper is concerned with a boundary value problem involving differential equations and boundary conditions of the form

$$
\begin{aligned}
&y_i' = [A_{ij}(x) + \lambda B_{ij}(x)]y_j, \\
(1.1)\quad &s_i[y, \lambda] \equiv (\overset{0}{M}_{ij} + \lambda \overset{1}{M}_{ij})y_j(a) + (\overset{0}{N}_{ij} + \lambda \overset{1}{N}_{ij})y_j(b) = 0 \\
&\qquad (a \leqq x \leqq b;\ i, j = 1, 2, \cdots, n),
\end{aligned}
$$

where the matrix of constants $\|M_{ij}^0+\lambda M_{ij}^1,\ N_{ij}^0+\lambda N_{ij}^1\|$ has rank n for all values of the characteristic parameter λ. In his dissertation the author [3][1] extended to such systems the concept of definite self-adjointness introduced by Bliss [2] for problems with boundary conditions independent of the parameter. Earlier, Bliss [1] had formulated a definition of definite self-adjoint systems in such a manner that systems of this type had infinitely many characteristic values. This property is in general no longer true for systems that are definitely self-adjoint in the modified sense of Bliss [2], and the analogous definition of Bobonis [3] is such that definitely self-adjoint systems (1.1) need not possess an infinitude of characteristic values. As shown in [3], however, for definitely self-adjoint systems (1.1) the characteristic values are all real and have indices equal to their multiplicities; moreover, such systems admit expansion theorems analogous to those obtained by Bliss [2].

It is the purpose of the present paper to consider a definitely self-adjoint system (1.1) which satisfies the additional condition that the matrix $\|B_{ij}(x)\|$ is of constant rank on the interval $a \leqq x \leqq b$. Such a system is shown to be equivalent to a boundary value problem associated with the second variation of a calculus of variations problem of the type considered by Reid [4], and the extremizing properties of the characteristic values of the equivalent problem lead to necessary and sufficient conditions for the given problem to have an infinitude of characteristic values. The methods of proof herein used are analogous to those employed by Reid [5] in establishing the corresponding results for definitely self-adjoint systems whose boundary conditions are independent of λ.

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[1] The numbers in brackets refer to the bibliography at the end of this paper.

2. Statement of the problem. In the present paper matrix notation will be used whenever possible. The subscripts i, j, k, α, will have the range $1, 2, \cdots, n$. Capital italic letters denote n-rowed square matrices, the element in the ith row and jth column being denoted by the same letter with the subscript ij. The vector (y_i) is denoted by the lower case letter y; Ay and yA represent the vectors $(A_{ij}y_j)$ and (y_jA_{ji}), respectively, where the repeated subscript j indicates a summation with respect to the subscript over the range $1, 2, \cdots, n$. The scalar product y_jz_j of the vectors y and z will be written yz. The transpose of the matrix A will be denoted by $\bar{A}$. Whenever the elements of the matrix A are differentiable functions, the matrix of derivatives is denoted by A'. For brevity, we shall also write $\mathcal{L}[y]$ and $\mathcal{M}[z]$ for the adjoint differential operators whose components $\mathcal{L}_i[y]$ and $\mathcal{M}_i[z]$ are given by

$$\mathcal{L}_i[y] = y_i' - A_{ij}(x)y_j, \qquad \mathcal{M}_i[z] = z_i' + z_jA_{ji}(x).$$

The elements of the matrices $\|A_{ij}(x)\|$, $\|B_{ij}(x)\|$ are real-valued and continuous functions in $a \leqq x \leqq b$ and the matrices $M(\lambda) = \|M_{ij}(\lambda)\| = \|M_{ij}^0 + \lambda M_{ij}^1\|$, $N(\lambda) = \|N_{ij}(\lambda)\| = \|N_{ij}^0 + \lambda N_{ij}^1\|$ are such that the elements of M^0, M^1, N^0, N^1 are real-valued and the $n \times 2n$ matrix $\|M_{ij}^0 + \lambda M_{ij}^1, N_{ij}^0 + \lambda N_{ij}^1\|$ has rank n for all values of the parameter λ. It is assumed that not all of the elements of M^1, N^1 are zero.

The system of equations with boundary conditions to be considered here is

$$(2.1)\qquad \begin{aligned} y' &= [A(x) + \lambda B(x)]y; \\ s[y, \lambda] &\equiv s^0[y] + \lambda s^1[y] \\ &\equiv (M^0 + \lambda M^1)y(a) + (N^0 + \lambda N^1)y(b) = 0. \end{aligned}$$

Throughout the present paper we assume that the boundary conditions of (2.1) satisfy the following hypothesis:

For all values of λ *the* $n \times 2n$ *matrix* $\|M(\lambda)\ N(\lambda)\|$ *has rank* n. *Moreover, there exist matrices* $P(\lambda) = P^0 + \lambda P^1$, $Q(\lambda) = Q^0 + \lambda Q^1$, *together with matrices* M^*, N^*, P^*, Q^* *independent of* λ *such that the* $2n \times 2n$ *matrices*

$$(2.2)\qquad \begin{Vmatrix} -P^* & -P(\lambda) \\ Q^* & Q(\lambda) \end{Vmatrix}, \qquad \begin{Vmatrix} M(\lambda) & N(\lambda) \\ M^* & N^* \end{Vmatrix}$$

are reciprocals for all values of λ.

In particular, the differential system adjoint to (2.1) is given by

$$(2.3)\qquad \begin{aligned} z' &= -z[A(x) + \lambda B(x)]; \\ t[z, \lambda] &\equiv t^0[z] + \lambda t^1[z] \equiv z(a)[P^0 + \lambda P^1] + z(b)[Q^0 + \lambda Q^1] = 0. \end{aligned}$$

The condition that the matrices (2.2) are reciprocals leads to the following useful relations

$$(2.4)\qquad \begin{aligned} -P^*M^0 - P^0M^* &= I, & Q^*M^0 + Q^0M^* &= 0,\\ -P^*M^1 - P^1M^* &= 0, & Q^*M^1 + Q^1M^* &= 0,\\ -P^*N^0 - P^0N^* &= 0, & Q^*N^1 + Q^1N^* &= 0,\\ -P^*N^1 - P^1N^* &= 0, & Q^*N^0 + Q^0N^* &= I. \end{aligned}$$

$$(2.5)\qquad M(\lambda)P(\lambda) - N(\lambda)Q(\lambda) = 0,$$

$$(2.6)\qquad -M^*P^0 + N^*Q^0 = I,$$

$$(2.7)\qquad -M^0P^* + N^0Q^* = I,$$

$$(2.8)\qquad -M^*P^* + N^*Q^* = 0,$$

$$(2.9)\qquad \begin{aligned} -M^*P^1 + N^*Q^1 &= 0,\\ -M^1P^* + N^1Q^* &= 0. \end{aligned}$$

In the above $I=\|I_{ij}\|$ and $0=\|0_{ij}\|$ denote the $n\times n$ identity and 0 matrices respectively.

Setting

$$s^*[y] = M^*y(a) + N^*y(b), \qquad t^*[z] = z(a)P^* + z(b)Q^*,$$

we have, in view of equations (2.4), that

$$t^*[z]s[y,\lambda] + t[z,\lambda]s^*[y] = z(x)y(x)\Big|_a^b.$$

Consequently

$$(2.10)\qquad t^*[z]s^0[y] + t^0[z]s^*[y] = z(x)y(x)\Big|_a^b,$$

$$(2.11)\qquad t^*[z]s^1[y] + t^1[z]s^*[y] = 0,$$

for arbitrary sets $y(a)$, $y(b)$, $z(a)$, $z(b)$.

It is to be emphasized that the boundary conditions of our problem are unchanged if the matrices $M(\lambda)$ and $N(\lambda)$ are replaced by $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$ where the matrix $\Gamma(\lambda)$ is nonsingular for all values of λ and such that the product matrices $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$ remain linear in λ.

The hypotheses under which the boundary value problem is to be developed are the following:

(H_1) The system (2.1) is self-adjoint under the nonsingular transformation $z=T(x)y$, where $T(x)$ is a nonsingular matrix with real-valued elements of class C'.

(H_2) $S(x) = \bar{T}(x)B(x)$ is symmetric on $a \leqq x \leqq b$.

(H_3) The matrix

$$\left\| \begin{array}{cc} \bar{T}(a)P^*M^1 & \bar{T}(a)P^*N^1 \\ \bar{T}(b)Q^*M^1 & \bar{T}(b)Q^*N^1 \end{array} \right\| \tag{2.12}$$

belonging to the quadratic form $Q[y(a), y(b)] = t^*[Ty]s^1[y]$ is symmetric.

(H_4) The quadratic expression

$$K[y] \equiv Q[y(a), y(b)] + \int_a^b ySydx \tag{2.13}$$

is non-negative for arbitrary vectors $y(x)$ whose components are continuous on $a \leqq x \leqq b$.

(H_5) For an arbitrary value of λ the only solution of the system (2.1) satisfying $K[y] = 0$ is $y(x) \equiv 0$ on $a \leqq x \leqq b$.

(H_6) $B(x)$ is of constant rank $n - m$, $0 \leqq m < n$, on $a \leqq x \leqq b$.

The characteristic values of problems satisfying hypotheses (H_1)–(H_5) are all real and the zeros of a permanently convergent power series (see Bobonis [3]), so that there are at most a countable number of such values. Moreover since hypotheses (H_1)–(H_6) remain invariant under a linear change of parameter, it may be assumed without loss of generality that $\lambda = 0$ is not a characteristic value of the problem considered.

Necessary and sufficient conditions for the system (2.1) to be self-adjoint under the transformation $z = Ty$ are

$$TA + \bar{A}T + T' = 0, \qquad TB + \bar{B}T = 0, \tag{2.14}$$

$$M(\lambda)T^{-1}(a)\bar{M}(\lambda) = N(\lambda)T^{-1}(b)\bar{N}(\lambda). \tag{2.15}$$

For the proof of the above conditions see Bobonis [3]. We have further that equations (2.5), (2.15), together with the hypothesis (H_3), justify the relations (Bobonis [3])

$$M(\lambda)T^{-1}(a) = C\bar{P}(\lambda), \qquad N(\lambda)T^{-1}(b) = C\bar{Q}(\lambda), \tag{2.16}$$

where C is a nonsingular constant matrix independent of λ.

We also have as a consequence of the previous hypotheses the following useful lemmas (Bobonis [3]).

Lemma 2.1. *Hypotheses* (H_2) *and* (H_4) *imply that the quadratic form* $yS(x)y$ *is positive semidefinite on* $a \leqq x \leqq b$.

Lemma 2.2. *Hypotheses* (H_3) *and* (H_4) *imply that the quadratic form* $Q[y(a), y(b)]$ *is positive semidefinite.*

LEMMA 2.3. *If for continuous* $y(x)$ *we have that* $K[y]=0$, *then hypotheses* (H_2), (H_3), *and* (H_4) *imply* $By\equiv 0$ *on* $a\leqq x\leqq b$, $s^1[y]=0$.

The previous lemmas enable us to state that if hypotheses (H_2), (H_3), (H_4) are satisfied, hypothesis (H_5) is equivalent to:

(H_5') The only solution of $y'=Ay$, $s^0[y]=0$ satisfying $ySy\equiv 0$ on $a\leqq x\leqq b$, $s^1[y]=0$, is $y(x)\equiv 0$.

We can also see with the help of relations (2.10) and (2.11) that if $y(x)$ and $y^*(x)$ are solutions of (2.1) for distinct values of λ, then

$$t^*[Ty]s^1[y^*]+\int_a^b ySy^*dx = 0. \tag{2.17}$$

As a consequence of hypothesis (H_6) it follows that there exist m sets of continuous functions $\Pi_{i\epsilon}(x)$ ($\epsilon=1, 2, \cdots, m$), such that on $a\leqq x\leqq b$,

$$S_{ij}(x)\Pi_{j\epsilon}(x) \equiv 0. \tag{2.18}$$

The sets $\Pi_{i\epsilon}$ can be chosen orthonormal in the sense that

$$\Pi_{i\epsilon}(x)\Pi_{i\sigma}(x) \equiv I_{\epsilon\sigma} \qquad (\epsilon, \sigma = 1, 2, \cdots, m).$$

In view of hypotheses (H_2) and (H_6) it follows that there exists an $n\times n$ matrix $R(x)$ such that the $(n+m)\times(n+m)$ matrices

$$\begin{Vmatrix} S_{ij}(x) & \Pi_{i\sigma}(x) \\ \Pi_{j\epsilon}(x) & 0_{\epsilon\sigma} \end{Vmatrix}, \qquad \begin{Vmatrix} R_{ij}(x) & \Pi_{i\sigma}(x) \\ \Pi_{j\epsilon}(x) & 0_{\epsilon\sigma} \end{Vmatrix}$$

are symmetric reciprocals on $a\leqq x\leqq b$.

Moreover, in view of Lemma 2.1 and relation (2.18), the quadratic form $R_{ij}(x)u_iu_j$ is positive for every non-null set (u_i) satisfying

$$\Pi_{j\epsilon}(x)u_j = 0 \qquad (\epsilon = 1, 2, \cdots, m).$$

The matrix

$$\begin{Vmatrix} P^*M^1T^{-1}(a) & P^*N^1T^{-1}(b) \\ Q^*M^1T^{-1}(a) & Q^*N^1T^{-1}(b) \end{Vmatrix} \tag{2.19}$$

is symmetric and positive semidefinite since it is obtained by multiplying the positive semidefinite matrix (2.12) on the left by the nonsingular matrix

$$\begin{Vmatrix} \overline{T}^{-1}(a) & 0 \\ 0 & \overline{T}^{-1}(b) \end{Vmatrix},$$

and by the transpose of the latter matrix on the right.

Multiplication of (2.19) on the left by the nonsingular matrix

$$\left\| \begin{matrix} -M^0 & N^0 \\ -M^* & N^* \end{matrix} \right\|,$$

and by the transpose of the latter matrix on the right, with the use of equations (2.7), (2.8), (2.9) and (2.16), show that the matrix $(-M^0P^1+N^0Q^1)\bar{C}$ is symmetric and positive semidefinite. Consequently, the matrix $E=C^{-1}(-M^0P^1+N^0Q^1)$ is also symmetric and positive semidefinite. Equations (2.5), (2.6), together with the second, fourth, sixth and seventh equations of (2.4), imply $P^1=P^*(-M^0P^1+N^0Q^1)=P^*CE$, $Q^1=Q^*(-M^0P^1+N^0Q^1)=Q^*CE$. As the $n\times 2n$ matrix $\|\bar{P}^*\bar{Q}^*\|$ is of rank n, it follows that $E_{ij}g_j=0$ is satisfied by a set (g_i) if and only if $P_{ij}{}^1g_j=0$, $Q_{ij}{}^1g_j=0$. In particular the rank of E is equal to the rank of the $n\times 2n$ matrix, $\|\bar{P}^1\,\bar{Q}^1\|$, which, in turn, is equal to the rank of $\|M^1\,N^1\|$ in view of (2.16). Hence the rank of E is equal to $n-r$, $0\leqq r<n$, and there exist r sets of constants $g_{i\nu}$ $(\nu=1, 2, \cdots, r)$ such that $E_{ij}g_{j\nu}=0$. The constants $g_{i\nu}$ can also be chosen normed and orthogonal in the sense that $g_{i\gamma}g_{i\nu}=I_{\gamma\nu}$. It clearly follows that there exists a constant matrix $\mathfrak{K}$ such that the symmetric $(n+r)\times(n+r)$ matrices

$$\left\| \begin{matrix} E_{ij} & g_{i\gamma} \\ g_{j\nu} & 0_{\nu\gamma} \end{matrix} \right\|, \qquad \left\| \begin{matrix} \mathfrak{K}_{ij} & g_{i\gamma} \\ g_{j\nu} & 0_{\nu\gamma} \end{matrix} \right\|$$

are reciprocals. Moreover, the quadratic form $\mathfrak{K}_{ij}u_iu_j$ is positive for all non-null sets (u_i) satisfying $g_{j\nu}u_j=0$ $(\nu=1, 2, \cdots, r)$.

3. A minimum problem and its accessory boundary value problem. An arc $z(x)$ is admissible if the functions $z_i(x)$ are of class D' on $a\leqq x\leqq b$ and satisfy the differential equations

$$\Pi_{j\epsilon}(x)\mathfrak{M}_j[z] = 0 \qquad (\epsilon = 1, 2, \cdots, m). \tag{3.1}$$

The class H_1^* is defined as the totality of admissible functions satisfying the following additional conditions

$$g_{i\gamma}\overset{0}{t_i}[z] = 0 \qquad (\gamma = 1, 2, \cdots, r), \tag{3.2}$$

$$G[z(a), z(b)] + \int_a^b zKzdx = 1, \tag{3.3}$$

where $G[z(a), z(b)]$ is a quadratic form in the arguments $z(a)$, $z(b)$ having (2.19) as matrix of coefficients, and $K=-B\bar{T}^{-1}$. The second equation (2.14) and the symmetry of S show that K is symmetric.

Suppose H_1^* is nonvacuous, and consider the problem of minimizing the expression

$$(3.4)\qquad I[z] = t^0[z]\mathcal{H}t^0[z] + \int_a^b \mathcal{M}[z]R\mathcal{M}[z]dx$$

in this class. In view of (3.2) the quadratic form $t^0[z]\mathcal{H}t^0[z]$ is positive unless $t^0[z]=0$; similarly, (3.1) implies that the integral of (3.4) is positive unless $\mathcal{M}[z]=0$ on $a\leqq x\leqq b$. As $\lambda=0$ is not a characteristic value of (2.1), it is also not a characteristic value of the adjoint system and consequently $I[z]>0$ for all arcs of H_1^*. For a minimizing arc $z(x)$ define

$$(3.5)\qquad R_{ij}\mathcal{M}_j[z] + \Pi_{i\epsilon}\mu_\epsilon = \zeta_i.$$

From the first necessary conditions of the above defined calculus of variations problem we know that there exist multipliers $\mu_\epsilon(x)$, Λ, and d_γ such that in addition to (3.1), (3.2), and (3.3) we have

$$(3.6)\qquad \zeta' - A\zeta + \Lambda Kz = 0,$$

$$(3.7)\qquad \begin{aligned} \overset{0}{P}_{ij}(\mathcal{H}_{jk}\overset{0}{t}_k[z] + g_{j\gamma}d_\gamma) - \Lambda\overset{*}{P}_{ij}\overset{1}{s}_j[\overset{-1}{T}z] - \zeta_i(a) &= 0,\\ \overset{0}{Q}_{ij}(\mathcal{H}_{jk}\overset{0}{t}_k[z] + g_{j\gamma}d_\gamma) - \Lambda\overset{*}{Q}_{ij}\overset{1}{s}_j[\overset{-1}{T}z] + \zeta_i(b) &= 0.\end{aligned}$$

Solving (3.1) and (3.5) simultaneously we have that

$$(3.8)\qquad \mathcal{M}[z] = S\zeta, \qquad \mu_\epsilon = \Pi_{j\epsilon}\zeta_j.$$

In view of (2.5), (2.6), (2.7), and (2.8) it follows that (3.7) is equivalent to

$$(3.9)\qquad s^0[\zeta] = \Lambda s^1[T^{-1}z],$$

$$(3.10)\qquad \mathcal{H}_{ij}\overset{0}{t}_j[z] + g_{i\gamma}d_\gamma = -\overset{*}{s}_i[\zeta].$$

Solution of (3.10) simultaneously with (3.2) and use of equation (2.6) together with the second, fourth, sixth, and seventh of equations (2.4) yield

$$(3.11)\qquad t^0[z] - C^{-1}s^1[\zeta] = 0, \qquad d_\gamma = -g_{i\gamma}\overset{*}{s}_i[\zeta].$$

Therefore the system (3.1), (3.2), (3.5), (3.6), and (3.7) is equivalent to the system

$$(3.12)\qquad \begin{aligned} \mathcal{M}[z] &= S\zeta, & t^0[z] - C^{-1}s^1[\zeta] &= 0,\\ \mathcal{L}[\zeta] &= -\Lambda Kz, & s^0[\zeta] - \Lambda s^1[T^{-1}z] &= 0.\end{aligned}$$

Concerning the above system we shall prove the following result.

THEOREM 3.1. *The system* (3.12) *is normal.*

Suppose the system is not normal. Then functions $z_i(x)\equiv 0$, $\zeta_i(x)\not\equiv 0$ exist satisfying (3.12) for constant Λ. The functions $\zeta_i=\Pi_{i\epsilon}\mu_\epsilon$ would then satisfy the system

$$\mathcal{L}[\zeta]=0, \qquad s^0[\zeta]=0, \qquad s^1[\zeta]=0.$$

Since $B_{ij}\Pi_{j\epsilon}\equiv 0$ we have that $B_{ij}\zeta_j\equiv 0$. This implies by hypothesis (H_5') that $\zeta_i(x)\equiv 0$ which is contrary to the hypothesis that $\zeta_i(x)\not\equiv 0$. Therefore, the system is normal.

The positiveness and reality of the characteristic values of the system (3.12) have been proved by Reid [4]. We also know that the characteristic values of such a system are at most denumerably infinite in number, since they are the zeros of a permanently convergent power series.

4. Sufficient conditions for the existence of infinitely many characteristic values.

To prove our sufficiency theorem use will be made of two theorems proved by Reid [4]. They will be inserted here for reference.

THEOREM 4.1. *Suppose that the class H_1^* is not empty and $\Lambda=\Lambda_1$ is the greatest lower bound of $I[z]$ in this class; then $\Lambda_1>0$ and $\Lambda=\Lambda_1$ is the least characteristic value of* (3.12).

THEOREM 4.2. *Suppose $\Lambda_1<\Lambda_2<\cdots<\Lambda_{t-1}$ are consecutive characteristic values of* (3.12) *and corresponding to $\Lambda=\Lambda_p$ $(p=1, 2, \cdots, t-1)$; there are r_p linearly independent solutions z_{iq_p}, ζ_{iq_p} $(q_p=1, 2, \cdots, r_p)$. Define the class H_t^* as the subclass of arcs belonging to H_1^* satisfying the conditons*

$$G[z(a), z_{q_p}(b)]+\int_a^b z_iK_{ij}z_{jq_p}dx=0.$$

Then, if H_t^* is not empty and Λ_t is the greatest lower bound of $I[z]$ in this class, $\Lambda_t>\Lambda_{t-1}$ and $\Lambda=\Lambda_t$ is a characteristic value of (3.12).

Consider once more system (3.12). Let z, ζ be a solution of this system for some value of Λ. If we use(2.14) and (2.16) it then follows that $z=(-1/\Lambda^{1/2})T\eta$ defines a vector η such that η, ζ is a solution of

$$(4.1)\qquad \begin{aligned} \mathcal{L}[\eta]&=\Lambda^{1/2}B\zeta, & s^0[\eta]+\Lambda^{1/2}s^1[\zeta]&=0,\\ \mathcal{L}[\zeta]&=\Lambda^{1/2}B\eta, & s^0[\zeta]+\Lambda^{1/2}s^1[\eta]&=0.\end{aligned}$$

Now, if η^*, ζ^* is a solution of (4.1) the functions $\eta=\eta^*+\zeta^*$, $\zeta=\eta^*+\zeta^*$ and $\eta=\eta^*-\zeta^*$, $\zeta=-\eta^*+\zeta^*$ are also solutions of this system. Hence if Λ is a characteristic value of (3.12) of index r, it follows that there exist r linearly independent solutions η, ζ of (4.1) such that for each of these solutions we have either $\eta=\zeta$ or $\eta=-\zeta$. Suppose that η, $\zeta=\eta$ is a solution of (4.1). Then $y=\eta$ is a solution of (2.1) for $\lambda=\Lambda^{1/2}$. Likewise, if η, $\zeta=-\eta$ is a solution of (4.1), $y=\eta$ is a solution of (2.1) for $\lambda=\Lambda^{1/2}$. Therefore, the sum of the indices of $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ as characteristic values of (2.1) is not smaller than the index of Λ as a characteristic value of (3.12). On the other hand, if $y(x)$ is a characteristic solution of (2.1) for some $\lambda\neq 0$, $z=(-1/\lambda)Ty$, $\zeta=y$ is a solution of (3.12) for $\Lambda=\lambda^2$. Relation (2.17) implies that the set of characteristic solutions corresponding to values $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ are linearly independent and, therefore, we have that the index of Λ as a characteristic value of (3.12) is not less than the sum of the indices of $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ as characteristic values of (2.1). Hence, we have the following result.

THEOREM 4.3. *If Λ is a characteristic value of* (3.12), *then either $\Lambda^{1/2}$ or $-\Lambda^{1/2}$ is a characteristic value of* (2.1); *conversely, if λ is a characteristic value of* (2.1), *then $\Lambda=\lambda^2$ is a characteristic value of* (3.12) *with index which equals the sum of the indices of λ and $-\lambda$ as characteristic values of* (2.1).

The three preceding theorems imply the analogue of Theorem 4.1 of Reid [5].

THEOREM 4.4. *A system* (2.1) *satisfying hypotheses* (H_1)–(H_6) *has an infinity of characteristic values if and only if there are infinitely many arcs $z=w_p(x)$ $(p=1, 2, \cdots)$ satisfying* (3.1) *and* (3.2) *and such that for each r and arbitrary constants $f_t\neq 0_t$ $(t=1, 2, \cdots, r)$ the arcs $w=w_t(x)f_t$ satisfy the condition*

$$G[w(a), w(b)] + \int_a^b wKwdx > 0.$$

Finally, it is to be noted that if z is admissible then there exists a vector $h(x)$ with components piecewise continuous on $a\leqq x\leqq b$ and such that $\mathfrak{M}[z]=-h(x)B(x)$. Furthermore, if the end values of z satisfy (3.2) it follows that there exists a constant vector $k\equiv(k_i)$ such that $t^0[z]=kE$. From the form of the matrix E it then follows that $h(a)=kC^{-1}M^0$, $h(b)=-kC^{-1}N^0$ satisfy $t^0[z]+t^1[h]=0$. That is, an admissible z satisfies (3.1) if and only if

$$(4.2)\qquad \mathfrak{M}[z] = -h(x)B(x), \qquad t^0[z]+t^1[h]=0,$$

where $h(x)$ is piecewise continuous on $a\leqq x\leqq b$, it being understood, in particular, that the components of $h(x)$ may be discontinuous at $x=a$ and $x=b$.

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ON THE SUMMATION OF MULTIPLE FOURIER SERIES. III[1]

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Let $f(x)=f(x_1, \cdots, x_k)$ be a function of the Lebesgue class L, which is periodic in each of the k-variables, having the period 2π. Let

$$a_{\nu_1\cdots\nu_k}=\frac{1}{(2\pi)^k}\cdot\int_{-\pi}^{+\pi}\cdots\int_{-\pi}^{+\pi} f(x)\exp\left\{-i(\nu_1x_1+\cdots+\nu_kx_k)\right\}dx_1\cdots dx_k,$$

where $\{\nu_k\}$ are all integers. Then the series $\sum a_{\nu_1}\cdots_{\nu_k}\exp i(\nu_1x_1+\cdots+\nu_kx_k)$ is called the multiple Fourier series of the function $f(x)$, and we write

$$f(x)\sim\sum a_{\nu_1\cdots\nu_k}\exp i(\nu_1x_1+\cdots+\nu_kx_k).$$

Let the numbers $(\nu_1^2+\cdots+\nu_k^2)$, when arranged in increasing order of magnitude, be denoted by $\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots$, and let

$$C_n(x)=\sum a_{\nu_1\cdots\nu_k}\exp i(\nu_1x_1+\cdots+\nu_kx_k),$$

where the sum is taken over all $\nu_1^2+\cdots+\nu_k^2=\lambda_n$,

$$\phi(x,t)=\sum C_n(x)\exp(-\lambda_n t),$$

$$S_R(x)=\sum_{\lambda_n\leqq R^2} C_n(x),\qquad \lambda_n\leqq R^2<\lambda_{n+1}.$$

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[1] Papers I and II with the same title are to appear in Proc. London Math. Soc.