ON k-TO-1 TRANSFORMATIONS

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The following results are extensions of certain of the theorems of O. G. Harrold (*Exactly* (k, 1) *transformations on connected linear graphs*, Amer. J. Math. vol. 62 (1940) pp. 823–834).

Let X and Y be compact Hausdorff spaces and let f be a continuous transformation of X onto Y. Let k be a positive integer and let μE denote the cardinal of the set E. We say that f is *at most k-to-1* (or *exactly k-to-1*) in case $y \in Y$ implies $\mu f^{-1}(y) \leqq k$ (or $\mu f^{-1}(y) = k$). Let $o(x)$ denote the order of the point x. That is to say, $o(x)$ is the smallest integer m such that μ bdy $U = m$ for an arbitrarily small open neighborhood U of x, if such exists; otherwise $o(x)$ is ∞.

THEOREM 1. *If f is at most k-to-1 and if the inverse points of $y \in Y$ are $x_1, \cdots, x_n$, then $\sum_{i=1}^{n} o(x_i) \leqq k \cdot o(y)$.*

PROOF. We may suppose $o(y)$ is finite. Let $U_1, \cdots, U_n$ be neighborhoods (open neighborhoods) of $x_1, \cdots, x_n$ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy $W = o(y)$ and $f^{-1}(W) \subset \bigcup_{i=1}^{n} U_i$. Define $V_i = U_i \cap f^{-1}(W)$. It follows that $k \cdot o(y) = k \cdot \mu$ bdy $W \geqq \mu f^{-1}(\text{bdy } W) \geqq \mu$ bdy $f^{-1}(W)$ $= \mu \bigcup_{i=1}^{n}$ bdy $V_i = \sum_{i=1}^{n} \mu$ bdy V_i. We conclude that each $o(x_i)$ is finite. By taking the U_i sufficiently small, μ bdy $V_i \geqq o(x_i)$. The conclusion follows.

COROLLARY 1. *If X and Y are continua and if f is exactly k-to-1, then each inverse point of an end point of Y is an end point of X.*

Let P denote the property of being a continuum in X on which f is exactly k-to-1.

THEOREM 2. *If X has property* P *irreducibly, then Y has no end point; if moreover $k = 2$, then Y has no cut point.*

PROOF. We prove the first statement. Suppose Y has an end point y. Write $f^{-1}(y) = \bigcup_{i=1}^{k} x_i$. Let $U_1, \cdots, U_k$ be neighborhoods of $x_1, \cdots, x_k$ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy $W = o(y) = 1$ and $f^{-1}(W) \subset \bigcup_{i=1}^{k} U_i$. Define $V_i = U_i \cap f^{-1}(W)$. As in the proof of Theorem 1 it follows that $k = k \cdot o(y) \geqq \sum_{i=1}^{k} \mu$ bdy V_i. Hence, each bdy V_i consists of a single point and it follows easily that $X - \bigcup_{i=1}^{k} V_i = X - f^{-1}(W)$ is a proper

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subcontinuum of X on which f is exactly k-to-1. This is a contradiction.

We prove the second statement. Suppose y is a cut point of Y and let $Y-y=Y_1\cup Y_2$ be a separation. Then, $X-f^{-1}(y)=f^{-1}(Y_1)\cup f^{-1}(Y_2)$ is a separation and since also $f^{-1}(y)$ consists of only two points, at least one of the sets $f^{-1}(Y_i)\cup f^{-1}(y)$ $(i=1, 2)$ is a continuum. This contradicts the hypothesis of irreducibility.

COROLLARY 2. *No dendrite is a continuous exactly k-to-1, $k>1$, image of a continuum.*

PROOF. Suppose $f(X)=Y$ is exactly k-to-1, $k>1$, where X is a continuum and Y is a dendrite. By use of Zorn's lemma it may readily be seen that there exists a subcontinuum X_0 of X which has property P irreducibly. The nondegenerate continuum $f(X_0)$ is a dendrite and hence has an end point. This is impossible by Theorem 2.

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