CURVATURE IN HERMITIAN METRIC

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Hermitian metric has the peculiarity of favoring negative curvature over positive curvature. We shall explain this phenomenon by pointing out that in the case of an isometric analytic imbedding the relative curvature is on the whole negative; also, by reduction to a *limiting case* of imbedding we shall explain why an invariant metric in the theory of Fuchsian groups is likely to be hyperbolic; see Hua [6].[1]

However, on the other hand, an Hermitian metric is very rigid, and the possibility of imbedding into a *finitely-dimensional* enveloping space is very remote. The classical conjectures about the possibility of Euclidean imbedding are rendered entirely false, but as a compensation, there are more and better theorems about equivalence and uniqueness.

1. **Hermitian metric.** There are many places in the literature where an introduction to the theory of Hermitian metric can be found. We shall refer to our own summary as given in Bochner [2, chap. 2]. We quote from there that in discussing an Hermitian metric in a space V_n of n complex variables $z_1, \cdots, z_n$, the basic variables are the $2n$ conjugate complex quantities

$$(1) \qquad z_1, \cdots, z_n; \bar{z}_1, \cdots, \bar{z}_n$$

which we shall also denote indifferently by

$$(2) \qquad t_1, t_2, \cdots, t_{2n}.$$

Italic indices run from 1 to $2n$, Greek indices from 1 to n, and starring an index will add to it the value n if it is not greater than n, and subtract n from it if it is not less than $n+1$. All scalars and components of tensors are power series in (1), and they are always *self-adjoint*, meaning that starring all indices in a component of a tensor will change its value into its conjugate complex. Scalars are real-valued.

The fundamental tensor g_{ij} has the properties

$$g_{ij} = g_{ji}, \qquad g_{\alpha\beta} = g_{\alpha^*\beta^*} = 0$$

in addition to the self-adjointness property $g_{\alpha\beta^*} = \bar{g}_{\beta\alpha^*}$; also $g_{\alpha\beta^*}$ is positive-definite. In particular, we have

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

(3) $$ds^2 = g_{ij}dt_i dt_j = 2g_{\alpha\beta^*}dz_\alpha d\bar{z}_\beta.$$

Finally, we have Kaehler's restriction (non-torsion)

(4) $$\frac{\partial g_{\alpha\gamma^*}}{\partial z_\beta} = \frac{\partial g_{\beta\gamma^*}}{\partial z_\alpha}$$

which is equivalent with

(5) $$g_{\alpha\beta^*} = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}$$

for some scalar function $\Phi(z, \bar{z})$, locally. On replacing Φ by $(\Phi+\bar{\Phi})/2$ we can always make it real-valued. If we form Γ^i_{jk} by the usual formula, then only the unmixed components $\Gamma^\alpha_{\beta\gamma}$ and their conjugates can be not equal to 0, and

(6) $$\Gamma^\alpha_{\beta\gamma} = g^{\alpha\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\gamma}.$$

If we form the curvature tensor R_{ijkl} by the usual formula, then it satisfies all classical relations. Furthermore, only the components of the form $R_{\alpha\beta^*\gamma\delta^*}$, $R_{\alpha\beta^*\gamma^*\delta}$, $R_{\alpha^*\beta\gamma\delta^*}$, $R_{\alpha^*\beta\gamma^*\delta}$ can be different from 0. Finally, we have

$$R^\alpha_{\beta\gamma^*\delta} = \frac{\partial}{\partial \bar{z}_\gamma}\Gamma^\alpha_{\beta\delta}$$

and therefore

$$\begin{aligned} R_{\alpha^*\beta\gamma^*\delta} &= g_{\alpha^*\rho}\frac{\partial}{\partial \bar{z}_\gamma}\left(g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\right) \\ &= \frac{\partial}{\partial \bar{z}_\gamma}\left(g_{\alpha^*\rho}g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\right) - g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\frac{\partial g_{\alpha^*\rho}}{\partial \bar{z}_\gamma}. \end{aligned}$$

Hence we have finally

(7) $$R_{\alpha\beta^*\gamma\delta^*} = \frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar{z}_\delta} - g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial \bar{z}_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma}.$$

In particular we have the symmetry relations

(8) $$R_{\alpha\beta^*\gamma\delta^*} = R_{\gamma\beta^*\alpha\delta^*} = R_{\alpha\delta^*\gamma\beta^*}$$

which are unmatched in the case of real variables.

THEOREM 1. *If two tensors g_{ij} of our description in the same coordinates have the same curvature tensor $R_{\alpha\beta^*\gamma\delta^*}$, then they are equal to within an allowable transformation.*

The conclusion also holds if the components of the type $R_{\alpha}{}^{\beta}{}_{\gamma\delta^*}$ or of some other type are assumed to be the same.

PROOF. We take one of the tensors g_{ij} and we introduce the power series

$$\Phi(z_1, \cdots, z_n; \bar{z}_1, \cdots, \bar{z}_n) \tag{9}$$

for which (5) holds. We can obviously omit from (9) all monomials

$$a z_1^{p_1} \cdots z_n^{p_n} \bar{z}_1^{q_1} \cdots \bar{z}_n^{q_n} \tag{10}$$

for which either $p_1 + \cdots + p_n = 0$ or $q_1 + \cdots + q_n = 0$, and thus assume (9) in the form

$$a_{\alpha\beta^*} z_\alpha \bar{z}_\beta + \Phi_3(z, \bar{z}) \tag{11}$$

where $a_{\alpha\beta^*} = g_{\alpha\beta^*}(0)$, and Φ_3 contains only monomials with

$$p_1 + \cdots + p_n + q_1 + \cdots + q_n \geqq 3.$$

An allowable transformation will carry (11) into

$$\sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha + \Phi_3(z, \bar{z}), \tag{12}$$

and on making it real, whenever (12) contains a monomial (10) it also contains $\bar{a} \bar{z}_1^{p_1} \cdots \bar{z}_n^{p_n} z_1^{q_1} \cdots z_n^{q_n}$. Therefore, we can put

$$\Phi_3 = \sum_{\alpha=1}^{n} \left(z_\alpha \overline{f_\alpha(z)} + \bar{z}_\alpha f_\alpha(z)\right) + \Phi_{2,2}(z, \bar{z}),$$

where each function $f_1, \cdots, f_n(z)$ is a power series in $z_1, \cdots, z_n$ with terms of total degree not less than 2, and $\Phi_{2,2}$ contains only terms with

$$p_1 + \cdots + p_n \geqq 2 \quad \text{and} \quad q_1 + \cdots + q_n \geqq 2.$$

After the allowable transformation $z_\alpha' = z_\alpha + f_\alpha(z)$, our function has the form

$$\Phi = \sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha + \psi_{2,2}(z, \bar{z}). \tag{13}$$

If now we introduce the derivatives

$$\frac{\partial^{p_1 + \cdots + p_n + q_1 + \cdots + q_n} \Phi}{\partial z_1^{p_1} \cdots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \cdots \partial \bar{z}_n^{q_n}}$$

at the origin, then our normalization has fixed their values in case the total order is 1, 2 or 3; and for the total order 4, 5, 6, $\cdots$ if either $p_1+\cdots+p_n\leqq 1$ or $q_1+\cdots+q_n\leqq 1$. Now, if we write equation (7) in the form

$$\frac{\partial^4\Phi}{\partial z_\alpha\partial\bar z_\beta\partial z_\gamma\partial\bar z_\delta} = g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial\bar z_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} + R_{\alpha\beta^*\ \delta^*}, \tag{14}$$

and if we remember that each component $g^{\rho\sigma^*}$ is a rational function of the components $g_{\alpha\beta^*}$, then the missing derivatives of order 4 can be computed directly from (14); furthermore, the missing derivatives of higher order can be successively computed by taking all possible successive derivatives of (14). Also these computations are unique, which proves our contention. If the components $R_\alpha{}^\beta{}_{\gamma\delta^*}$ are given, the same conclusion will follow from the equations

$$\frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma\partial\bar z_\delta} = g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial\bar z_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} + g_{\beta^*\epsilon}R^{\epsilon}_{\alpha\gamma\delta^*},$$

and this completes the proof of Theorem 1.

THEOREM 2. *If the metric tensors g_{ij} and h_{ij} have the same geodesics, then their affine connections Γ^l_{ij} and Δ^l_{ij} are the same. Thus a projective collineation is automatically an affine collineation.*

PROOF. As in the real case we have, see [3, p. 132],

$$\overset{l}{\Delta}_{ij} - \overset{l}{\Gamma}_{ij} = \overset{l}{\delta}_i\psi_j + \overset{l}{\delta}_j\psi_i.$$

Putting $l=\alpha$, $i=\alpha$, $j=\gamma^*$, we obtain $\psi_{\gamma^*}=0$, and similarly $\psi_\gamma=0$.

2. **Sectional curvature.** A two-dimensional surface element through the origin is given in the form

$$t_i = \lambda^i x + \mu^i y, \tag{15}$$

that is

$$z_\alpha = \lambda^\alpha x + \mu^\alpha y, \qquad \bar z_\alpha = \overline{\lambda^\alpha}x + \overline{\mu^\alpha}y, \tag{16}$$

where x and y are real parameters, and the systems of complex numbers $\{\lambda^\alpha\}$ and $\{\mu^\alpha\}$ are nonproportional. If we make the decomposition

$$z_\alpha = x_\alpha + (-1)^{1/2}y_\alpha,\quad \lambda^\alpha = \overset{\alpha}{\lambda_0} + (-1)^{1/2}\overset{\alpha}{\lambda_1},\quad \mu^\alpha = \overset{\alpha}{\mu_0} + (-1)^{1/2}\overset{\alpha}{\mu_1},$$

then we can also write

$$x_\alpha = \overset{\alpha}{\lambda_0}x + \overset{\alpha}{\mu_0}y, \qquad y_\alpha = \overset{\alpha}{\lambda_1}x + \overset{\alpha}{\mu_1}y. \tag{17}$$

We now set up the number

$$K = \frac{R_{hijk}\lambda^h\mu^i\lambda^j\mu^k}{(g_{hj}g_{ik} - g_{hk}g_{ij})\lambda^h\mu^i\lambda^j\mu^k} \tag{18}$$

and we claim that it is the real-valued curvature for the surface element (17) and the line element (3). In fact, if we denote the real component y_α by $x_{\alpha+n}$, if we write the line element (3) in the real form

$$h_{ij}(x)dx_i dx_j,$$

if we introduce the ordinary curvature tensor corresponding to the latter line element, if we form with that curvature tensor the sectional curvature for the section (17), and if finally we carry out purely formally the transformation of coordinates

$$z_\alpha = x_\alpha + (-1)^{1/2}x_{\alpha+n}, \qquad \bar{z}_\alpha = x_\alpha - (-1)^{1/2}x_{\alpha+n}$$

as if it were an allowable transformation; then by formal properties of invariance the curvature will appear in the form (18).

THEOREM 3. *For $n>1$, if at every point the sectional curvature K is the same for all possible two-dimensional sections, then the curvature tensor is identically zero.*

PROOF. Relation

$$[R_{hijk} - K(g_{hj}g_{ik} - g_{hk}g_{ij})]\lambda^h\mu^i\lambda^j\mu^k = 0 \tag{19}$$

is fulfilled for $2n$ independent variables λ^α, μ^α and their conjugate values $\overline{\lambda^\alpha}$, $\overline{\mu^\alpha}$. We are now applying a fundamental lemma to the effect that whenever a power series

$$\Phi(u_1, \cdots, u_m; v_1, \cdots, v_m)$$

is zero identically in $u_1, \cdots, u_m$ for $v_k = \bar{u}_k$, $k=1, \cdots, n$, then it is identically 0 in the independent variables u_k, v_k. If we apply this to (19) then for K independent of λ and μ, we obtain by a classical formal procedure the relation

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}). \tag{20}$$

In particular, we obtain

$$R_{\alpha\beta^*\gamma\delta^*} = -Kg_{\alpha\delta^*}g_{\beta^*\gamma},$$

and on applying the important relation (8) we deduce

$$Kg_{\alpha\delta^*}g_{\beta^*\gamma} = Kg_{\gamma\delta^*}g_{\beta^*\alpha}.$$

If we multiply both sides by $g^{\alpha\delta^*}g^{\beta^*\gamma}$ and contract we obtain $n^2K=K$, and hence $K=0$ for $n>1$.

DEFINITION. We call a section *holomorphic* if it is tangent to an analytically imbedded complex "curve." It is not hard to see that (16) is holomorphic if an only if there exists a *non-real* number ϕ such that $\mu^\alpha = \phi\lambda^\alpha$. If we transform the two real parameters x, y by a suitable nonsingular affine transformation $x' = \phi_1 x + \phi_2 y$, $y' = \phi_3 x + \phi_4 y$ with real coefficients ϕ_1, ϕ_2, ϕ_3, ϕ_4, then we can obtain the normalization

$$\mu^\alpha = (-1)^{1/2}\lambda^\alpha; \tag{21}$$

that is, the section can be written in the form

$$z_\alpha = \lambda^\alpha z; \qquad z = x + (-1)^{1/2} y. \tag{22}$$

THEOREM 4. *For a holomorphic section* (21) *we have*

$$\begin{aligned} K &= -\frac{R_{\alpha\beta^*\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}{g_{\alpha\beta^*}g_{\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}} \\ &= -2\,\frac{R_{\alpha\beta^*\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}{(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*})\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}. \end{aligned} \tag{23}$$

PROOF. This follows from the fact that for an arbitrary section (16) we have

$$R_{hijk}\lambda^h\mu^i\lambda^j\mu^k = R_{\alpha\beta^*\gamma\delta^*}(\lambda^\alpha\mu^{\beta^*} - \lambda^{\beta^*}\mu^\alpha)(\lambda^\gamma\mu^{\delta^*} - \lambda^{\delta^*}\mu^\gamma) \tag{24}$$

and

$$\begin{aligned} &(g_{hj}g_{ik} - g_{hk}g_{ij})\lambda^h\mu^i\lambda^j\mu^k \\ &\quad = g_{hj}\lambda^h\lambda^j g_{ik}\mu^i\mu^k - g_{hk}\lambda^h\mu^k g_{ij}\mu^i\lambda^j \\ &\quad = g_{\alpha\beta^*}g_{\gamma\delta^*}[(\lambda^\alpha\mu^\gamma - \lambda^\gamma\mu^\alpha)(\lambda^{\beta^*}\mu^{\delta^*} - \lambda^{\delta^*}\mu^{\beta^*}) \\ &\qquad + (\lambda^\alpha\mu^{\delta^*} - \lambda^{\delta^*}\mu^\alpha)(\lambda^{\beta^*}\mu^\gamma - \lambda^\gamma\mu^{\beta^*})]. \end{aligned} \tag{25}$$

We shall now draw an interesting conclusion.

THEOREM 5. *If at a point all holomorphic sections have the same curvature $K = b$, then at that point we have*

$$R_{\alpha\beta^*\gamma\delta^*} = -\frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*}). \tag{26}$$

Also, if (26) *holds at every point, then b is a constant.*

PROOF. By assumption we have

$$\left[R_{\alpha\beta^*\gamma\delta^*} + \frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\beta^*\gamma}g_{\alpha\delta^*})\right]\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*} = 0.$$

At first the relation holds whenever λ^{α^*} is conjugate complex to λ^{α}. By our previous argument it holds for independent variables $(\lambda^{\alpha}, \lambda^{\beta^*})$, and relation (26) follows now from the fact that both sides of it are symmetric in the pairs of indices (α, γ) and (β^*, δ^*).

Next, Bianchi's relation specializes to

$$R_{\alpha\beta^*\gamma\delta^*,\epsilon} = R_{\alpha\beta^*\epsilon\delta^*,\gamma}, \tag{27}$$

and if (26) holds at all points we obtain

$$b_{,\epsilon}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*}) = b_{,\gamma}(g_{\alpha\beta^*}g_{\epsilon\delta^*} + g_{\epsilon\beta^*}g_{\alpha\delta^*}).$$

This shows that $b(z, \bar{z})$ is independent of the variables z_α. The independence of $\bar{z}_\alpha$ follows from $R_{\alpha\beta^*\gamma\delta^*,\epsilon^*} = R_{\alpha\beta^*\gamma\epsilon^*,\delta^*}$.

3. **Fubini spaces.** For arbitrary real b we put

$$\Phi = \frac{2}{b} \log \left(1 + \frac{b}{2} \sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha\right) \equiv \frac{2}{b} \log S. \tag{28}$$

Therefore,

$$g_{\alpha\beta^*} = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\delta_{\alpha\beta^*}}{S} - \frac{b}{2} \frac{\bar{z}_\alpha z_\beta}{S^2}, \tag{29}$$

$$\frac{ds^2}{2} = \frac{\sum_\alpha | dz_\alpha |^2 + \frac{b}{2}\left(\sum_\alpha | z_\alpha |^2 \sum_\beta | dz_\beta |^2 - \left| \sum_\alpha \bar{z}_\alpha dz_\alpha \right|^2 \right)}{\left(1 + \frac{b}{2} \sum_\alpha | z_\alpha |^2\right)^2}. \tag{30}$$

At the origin we have $g_{\alpha\beta^*} = \delta_{\alpha\beta^*}$ and on computing (7) we obtain

$$R_{\alpha\beta^*\gamma\delta^*} = -\frac{b}{2}(\delta_{\alpha\beta^*}\delta_{\gamma\delta^*} + \delta_{\alpha\delta^*}\delta_{\beta\gamma^*}).$$

Thus (26) holds at the origin. The validity at other points follows from the existence of a transitive group of analytic homeomorphisms which leave the line element (3) invariant, see Fubini [4].

THEOREM 6. *For given b all spaces of constant holomorphic curvature b are equivalent.*

PROOF. The conclusion follows from

$$\frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar{z}_\delta} = g^{\rho\sigma^*} \frac{\partial g_{\rho\beta^*}}{\partial \bar{z}_\delta} \frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} - \frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\alpha\delta^*}g_{\gamma\beta^*})$$

as in the case of Theorem 1.

THEOREM 7. *For the Fubini space $b=1$ the general sectional curvature K lies in the interval*

$$1/4 \leqq K \leqq 1. \tag{31}$$

The maximum value $K=1$ is reached only if the section is holomorphic and the minimum value $K=1/4$ is reached, among others, if the section is totally real, that is

$$x_\alpha = \overset{\alpha}{\lambda}_0 x + \overset{\alpha}{\mu}_0 y, \qquad y_\alpha = 0; \qquad \alpha = 1, \cdots, n. \tag{32}$$

PROOF. At the origin, if we use (24) and (25) and if we put

$$Q(\lambda, \mu) = \sum_\alpha \lambda^\alpha \mu^{\alpha^*},$$

we obtain for K the value

$$\frac{Q(\lambda, \lambda)Q(\mu, \mu) + Q(\lambda, \mu)Q(\mu, \lambda) - Q(\lambda, \mu)^2 - Q(\mu, \lambda)^2}{4Q(\lambda, \lambda)Q(\mu, \mu) - 2Q(\lambda, \mu)Q(\mu, \lambda) - Q(\lambda, \mu)^2 - Q(\mu, \lambda)^2}.$$

If we introduce the quotient

$$t = \frac{Q(\lambda, \mu)}{(Q(\lambda, \lambda)Q(\mu, \mu))^{1/2}} = re^{i\psi},$$

then K has the value

$$\frac{1 + r^2 - 2r^2 \cos 2\psi}{4 - 2r^2 - 2r^2 \cos 2\psi} = 1 - \frac{3}{4} \frac{1 - r^2}{1 - ((1 + \cos 2\psi)/2)r^2}. \tag{33}$$

We always have $r \leqq 1$. The extreme value $r=1$ can only be reached if the section is holomorphic, and then (33) has the value 1. For $r<1$, the minimum is reached if t is real-valued, thus if $\sum \lambda^\alpha \mu^{\alpha^*}$ is real-valued. This will in particular occur for a section (32).

The Ricci tensor is defined by

$$R_{ij} = g^{pq} R_{pijq}, \qquad R_{\alpha\beta^*} = g^{\rho^*\sigma} R_{\alpha\rho^*\sigma\beta^*},$$

and it has the important property [2, formula (58)]

$$R_{\alpha\beta^*} = \frac{\partial^2 \log G}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{34}$$

where G is the determinant of the matrix $g_{\alpha\beta^*}$. The Ricci curvature in a given direction λ^i is defined as

$$\kappa = -\frac{R_{ij}\lambda^i\lambda^j}{g_{ij}\lambda^i\lambda^j} = -\frac{R_{\alpha\beta^*}\lambda^\alpha\lambda^{\beta^*}}{g_{\alpha\beta^*}\lambda^\alpha\lambda^{\beta^*}}.$$

THEOREM 8. *For a Fubini space we have*

$$R_{\alpha\beta^*} = -\frac{b}{2}(n+1)g_{\alpha\beta^*};$$

and the average Ricci curvature

$$\frac{\kappa}{2n-1} = b\,\frac{n+1}{4n-4}$$

which is everywhere constant, decreases monotonely towards the smallest possible value $1/4$, *as* $n\to\infty$.

The proof is quite obvious.

THEOREM 9. *If for a line element we have*

$$R_{\alpha\beta^*} = b(z,\bar{z})g_{\alpha\beta^*}, \tag{35}$$

then b is constant. In other words, if one of our spaces is an Einstein space at each point, it is so universally.

PROOF. By (34) we have $\partial R_{\alpha\beta^*}/\partial\bar{z}_\gamma = \partial R_{\alpha\gamma^*}/\partial\bar{z}_\beta$. Hence we obtain

$$b_{,\gamma^*}g_{\alpha\beta^*} + b\,\frac{\partial g_{\alpha\beta^*}}{\partial\bar{z}_\gamma} = b_{,\beta^*}g_{\alpha\gamma^*} + b\,\frac{\partial g_{\alpha\gamma^*}}{\partial\bar{z}_\beta}$$

and therefore $b_{,\gamma^*}g_{\alpha\beta^*} = b_{,\beta^*}g_{\alpha\gamma^*}$. This implies $b_{,\gamma^*}=0$ and similarly we obtain $b_{,\gamma}=0$.

For real spaces of constant curvature there exist modified line elements in terms of normal coordinates. In order to justify the absence of an analogue for Fubini spaces we shall point out a very general theorem.

THEOREM 10. *For a line element of our description there exist no allowable normal coordinates except in the obvious case of a flat space.*

PROOF. Normal coordinates have the usual consequences. In particular, we have

$$g_{ij}(t)t_j = g_{ij}(0)t_j$$

that is

$$g_{\alpha\beta^*}(z,\bar{z})\bar{z}_\beta = g_{\alpha\beta^*}(0)\bar{z}_\beta,$$

or

$$\frac{\partial^2\Phi}{\partial z_\alpha\partial\bar{z}_\beta}\bar{z}_\beta = g_{\alpha\beta^*}(0)\bar{z}_\beta.$$

For the function

$$\psi(z, \bar{z}) = \Phi - g_{\alpha\beta^*}(0) z_\alpha \bar{z}_\beta$$

we then obtain

$$\frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta} \bar{z}_\beta = 0, \qquad \alpha = 1, \cdots, n,$$

or

$$\frac{\partial \psi}{\partial \bar{z}_\beta} \bar{z}_\beta = F(\bar{z}).$$

On putting $\bar{z}_\beta = \bar{a}_\beta \zeta$, $H(z, \bar{a}, \zeta) \equiv \psi(z, \bar{a}\,\zeta)$, we then obtain $\zeta \partial H/\partial \zeta = F(\bar{a}\zeta)$; and since $F(\bar{z})$ has no constant term we obtain

$$H(z, \bar{a}, \zeta) = R(z, \bar{a}) + S(\bar{a}\zeta).$$

Therefore

$$\psi(z, \bar{z}) = R(z, \bar{z}/\zeta) + S(\bar{z}).$$

But the left side is independent of ζ, and so we have finally

$$\psi(z, \bar{z}) = R(z) + S(\bar{z})$$

and thus $\partial\psi/\partial z_\alpha \partial \bar{z}_\beta = 0$. Thus $g_{\alpha\beta^*}(z, \bar{z}) \equiv g_{\alpha\beta^*}(0)$, and our line element is flat.

However, as was shown in the proof of Theorem 1, there always exists a coordinate system which is *geodesic* at a prescribed point; meaning that the derivatives $\partial g_{\alpha\beta^*}/\partial z_\gamma$ and therefore also the coefficients $\Gamma^\alpha_{\beta\gamma}$ vanish at the point.

4. **Imbedding.** In addition to our space V_n with the metric (3) we consider a space V_m, $m > n$, in the complex variables $w_1, \cdots, w_m$, $m > n$, with a metric

$$2h_{\xi\eta^*}(w, \bar{w}) dw_\xi d\overline{w}_\eta \tag{36}$$

where $\xi, \eta, \zeta, \chi = 1, 2, \cdots, m$; and we assume that there exists a complex-analytic transformation

$$w_\xi = f^\xi(z_1, \cdots, z_n), \qquad \xi = 1, \cdots, m, \tag{37}$$

which is an *isometric* map of V_n into V_m. Thus

$$g_{\alpha\beta^*} = h_{\xi\eta^*} f^\xi{}_{,\alpha} \overline{f^\eta{}_{,\beta}} \tag{38}$$

where $f^\xi{}_{,\alpha} = \partial f^\xi / \partial z_\alpha$.

THEOREM 11. *If we denote the curvature tensor of V_n by $R_{\alpha\beta^*\gamma\delta^*}$ and of V_m by $S_{\xi\eta^*\zeta\chi^*}$, then we have*

(39) $$R_{\alpha\beta^*\gamma\delta^*} = S_{\xi\eta^*\zeta\chi^*}f^{\xi}{}_{,\alpha}\overline{f^{\eta}{}_{,\beta}}f^{\zeta}{}_{,\gamma}\overline{f^{\chi}{}_{,\delta}} + h_{\xi\eta^*}f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}}$$

where $f^{\xi}{}_{,\alpha,\gamma}$ denotes the second covariant derivative of $f^{\xi}(z)$ relative to the metric (3).

PROOF. We write (7) in the form

(40) $$R_{\alpha\beta^*\gamma\delta^*} = \frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar{z}_\delta} - \Gamma^{\rho}_{\alpha\gamma}\frac{\partial g_{\rho\beta^*}}{\partial \bar{z}_\delta}$$

and similarly

(41) $$S_{\xi\eta^*\zeta\chi^*} = \frac{\partial^2 h_{\xi\eta^*}}{\partial w_\zeta \partial \bar{w}_\chi} - \Delta^{\phi}_{\xi\zeta}\frac{\partial h_{\phi\eta^*}}{\partial \bar{w}_\chi},$$

where $\Delta^{\phi}_{\xi\zeta}$ is the affine connection pertaining to (36). Take a point z^0 and its image w^0. It can easily be seen that it will suffice to prove (39) under the assumption that the z-coordinates are geodesic at z^0 and the w-coordinates are geodesic at w^0. In this case all quantities

$$\frac{\partial g_{\alpha\beta^*}}{\partial z_\gamma}, \qquad \Gamma^{\alpha}_{\beta\gamma}, \qquad \frac{\partial h_{\xi\eta^*}}{\partial w_\zeta}, \qquad \Delta^{\xi}_{\eta\zeta}$$

and their conjugates are 0, and we easily obtain (39) by substituting (38) in the right side of (40) and carrying out the differentiation.

We now take a fixed point z^0 in V_n, and we introduce an arbitrary surface element

(42) $$z_\alpha - \overset{0}{z}_\alpha = \overset{\alpha}{\lambda} x + \overset{\alpha}{\mu} y$$

and its image

(43) $$w_\xi - \overset{0}{w}_\xi = \overset{\xi}{L} x + \overset{\eta}{M} y,$$

where

(44) $$L^{\xi} = f^{\xi}{}_{,\alpha}\lambda^{\alpha}, \qquad M^{\xi} = f^{\xi}{}_{,\alpha}\mu^{\alpha}.$$

If we multiply both sides of (39) by

(45) $$(\lambda^{\alpha}\mu^{\beta^*} - \lambda^{\beta^*}\mu^{\alpha})(\lambda^{\gamma}\mu^{\delta^*} - \lambda^{\delta^*}\mu^{\gamma})$$

and add up over α, β, γ, δ we obtain the following theorem.

THEOREM 12. *The sectional curvature*

(46) $$\frac{R_{abcd}\lambda^a\mu^b\lambda^c\mu^d}{(g_{ac}g_{bd} - g_{ad}g_{bc})\lambda^a\mu^b\lambda^c\mu^d}$$

in V_n is the sum of the corresponding sectional curvature

$$\frac{S_{ijkl}L^iM^jL^kM^l}{(h_{ik}h_{jl} - h_{il}h_{jk})L^iM^jL^kM^l} \tag{47}$$

in V_m and of the relative curvature

$$\frac{h_{\xi\eta^*}f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}}(\lambda^{\alpha}\mu^{\beta^*} - \lambda^{\beta^*}\mu^{\alpha})(\lambda^{\gamma}\mu^{\delta^*} - \lambda^{\delta^*}\mu^{\gamma})}{(g_{ac}g_{bd} - g_{ad}g_{bc})\lambda^a\mu^b\lambda^c\mu^d}. \tag{48}$$

For a holomorphic section (22) *the relative curvature has the value*

$$-\frac{h_{\xi\eta^*}f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\lambda^{\gamma}\overline{f^{\eta}{}_{,\beta,\delta}}\lambda^{\beta^*}\lambda^{\delta^*}}{(g_{\alpha\beta^*}\lambda^{\alpha}\lambda^{\beta^*})^2}. \tag{49}$$

It is therefore negative or 0, *and it has the value* 0 *only if the section is such that*

$$f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\lambda^{\gamma} = 0, \qquad \xi = 1, \cdots, m. \tag{50}$$

Condition (50) *formally corresponds to directions of asymptotic lines in the real case.*

For the Ricci tensor we have

$$R_{\alpha\delta^*} = S_{\xi\eta^*}f^{\xi}{}_{,\alpha}\overline{f^{\eta}{}_{,\delta}} + h_{\xi\eta^*}g^{\beta^*\gamma}f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}} \tag{51}$$

and the Ricci curvature $-R_{ad}\lambda^a\lambda^d/g_{ad}\lambda^a\lambda^d$ is the sum of $-S_{ij}L^iL^j/h_{ij}L^iL^j$ and of the relative Ricci curvature

$$-\frac{h_{\xi\eta^*}g^{\beta^*\gamma}f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\overline{f^{\eta}{}_{,\beta,\delta}}\lambda^{\delta^*}}{g_{\alpha\delta^*}\lambda^{\alpha}\lambda^{\delta^*}}. \tag{52}$$

The last term is negative or 0, *and it is* 0 *if and only if*

$$f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha} = 0, \qquad \xi = 1, \cdots, m; \gamma = 1, \cdots, n, \tag{53}$$

which is a stronger condition than (50).

If V_m has the Euclidean metric

$$dw_1 d\bar{w}_1 + \cdots + dw_m \overline{dw_m}, \tag{54}$$

then (48) is the entire amount of the sectional curvature of V_n. Now, put $m=3$, $n=2$, and

$$\overset{1}{f}(z) = z_1, \qquad \overset{2}{f}(z) = z_2, \qquad \overset{3}{f}(z) = 2^{-1}(z_1^2 + z_2^2).$$

The metric of V_2 is then $dz_1d\bar{z}_1 + dz_2d\bar{z}_2 + (z_1dz_1 + z_2dz_2)(\bar{z}_1d\bar{z}_1 + \bar{z}_2d\bar{z}_2)$. At the origin $z_1 = z_2 = 0$, the z-system is geodesic, and the numerator in (48) has at that point the value

$$\sum_{\alpha,\beta=1}^{3} (\lambda^\alpha \mu^{\beta^*} - \lambda^{\beta^*}\mu^\alpha)^2. \tag{55}$$

Since the denominator of (48) is always positive, see (25), the algebraic sign of (48) is that of (55). If, for instance, all components λ^α, μ^β are real, that is if our section is totally real, then (55) is positive, and thus we see that the relative sectional curvature may be positive for a nonholomorphic section.

THEOREM 13. *For $n=1$, if a line element*

$$ds^2 = g(z, \bar{z})dzd\bar{z} \tag{56}$$

of the special form

$$g(z, \bar{z}) = \sum_{p=0}^{\infty} a_p z^p \bar{z}^p \tag{57}$$

can be isometrically imbedded in V_m, with the Euclidean line element (54), *for some finite m, then the power series* (57) *is a finite polynomial in z, $\bar{z}$. In particular, the line element*

$$ds^2 = \frac{dzd\bar{z}}{(1 - z\bar{z})^2} = \sum_{p=0}^{\infty} (p + 1)z^p\bar{z}^p dzd\bar{z}, \tag{58}$$

although strictly hyperbolic, cannot be so imbedded.

For $n \geqq 1$, if in the line element

$$g_{\alpha\beta^*}(z, \bar{z})dz_\alpha d\bar{z}_\beta \tag{59}$$

in the power series of all $g_{\alpha\beta^*}$ only those monomials

$$z_1^{p_1} \cdots z_n^{p_n} \bar{z}_1^{q_1} \cdots \bar{z}_n^{q_n} \tag{60}$$

occur for which

$$p_1 + \cdots + p_n = q_1 + \cdots + q_n, \tag{61}$$

and if (59) *can be isometrically imbedded in a V_m, then all $g_{\alpha\beta^*}(z, \bar{z})$ are finite polynomials. In particular, the line elements arising in all types of matrix spaces*[2] *cannot be so imbedded.*

PROOF. For $n=1$, our imbedding means the existence of functions

$$w_k = f_k(z) = \sum_{p=1}^{\infty} c_{k,p}z^p, \qquad k = 1, \cdots, m, \tag{62}$$

such that

[2] See Siegel [7] and Hua [5] and [6].

$$\sum_k \left| \frac{df_k(z)}{dz} \right|^2 = \sum_p a_p z^p \bar{z}^p.$$

Therefore

(63) $$pq \sum_{k=1}^{m} c_{k,p} \overline{c_{k,q}} = \delta_{pq} a_{p-1}.$$

For each p, the system of numbers

$$C_p = \{c_{k,p}\}, \qquad k = 1, \cdots, m,$$

is a vector in complex m-space. By (63) any two of the vectors are orthogonal, and therefore all but at most m of them must be identically 0. Therefore at most m^2 among the numbers $c_{k,p}$ can be not equal to 0. Each $f_k(z)$ is a polynomial, and so is $g(z, \bar{z})$.

For $n \geqq 1$, if there exists an imbedding, then we write the mapping functions in the form

(64) $$f_k(z_\alpha) = \sum_{p=1}^{\infty} c_{k,p}(z_\alpha)$$

where $c_{k,p}$ is a homogeneous polynomial in the variables z_α of degree p, or identically zero. Now, put

(65) $$z_\alpha = \zeta_\alpha t, \qquad \bar{z}_\alpha = \bar{\zeta}_\alpha \bar{t},$$

where $\{\zeta_\alpha\}$ are arbitrary parameters in the neighborhood of the origin and t is the complex variable proper. Thus, (65) is a family of complex curves. For each one we derive from (59) a line element

$$g(\zeta, \bar{\zeta}; t, \bar{t}) dt d\bar{t},$$

where by our assumptions on (59),

$$g(\zeta, \bar{\zeta}; t, \bar{t}) = \sum_p a_p(\zeta, \bar{\zeta}) t^p \bar{t}^p.$$

On substituting (65) into (64), we obtain

$$f_k(\zeta_\alpha t) = \sum_p c_{k,p}(\zeta_\alpha) t^p,$$

and by our previous argument, corresponding to each special point $\{\zeta_\alpha^0\}$, only m^2 among the numbers $c_{k,p}(\zeta_\alpha^0)$ can be not equal to 0. Since each $c_{k,p}(\zeta)$ is a polynomial, it follows easily that all but m^2 among them must vanish, and thus $f_k(z_\alpha)$ is a polynomial. This proves the theorem.

Thus far we have been discussing imbedding into a V_m, for *finite m*.

The situation is radically different if we allow the imbedding space to become infinitely-dimensional. One possibility is to view certain constructions as limiting processes of ordinary imbedding; this will be our method of interpreting groups of automorphisms. However, another possibility is to actually perform an imbedding into a space of countably many variables $w_1, w_2, w_3, \cdots$ with a suitable metric

$$h_{\xi\eta^*}(w, \bar{w})dw_\xi \overline{dw_\eta}, \qquad \xi, \eta = 1, 2, 3, \cdots .$$

The most immediate line element would be the Hilbert metric

$$dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2 + \cdots .$$

In this case the hyperbolic line element (58) can very well be imbedded, namely by the transformation

$$w_k = z^k/(k+1)^{1/2}, \qquad k = 1, 2, \cdots .$$

Furthermore, our previous theorems remain valid, and thus for instance an elliptic line element cannot be isometrically imbedded in flat Hilbert space. Conceivably every finite-dimensional Hermitian line element without torsion could be imbedded into some universal countably-dimensional space with a fixed elliptic metric. It is interesting to note in this connection that the method of S. Bergman [1] for constructing an Hermitian metric on a domain of several complex variables consists precisely in constructing an analytic imbedding of the domain into such a universal enveloping space, the enveloping domain of Bergman being the countably-dimensional Fubini space with positive curvature $b=1$.

We shall not take up this problem in any way; however, we shall point out the following curious little theorem, which apparently is not true for finite m.

THEOREM 14. *Whenever a V_n can be isometrically imbedded in flat Hilbert space it can also be isometrically imbedded in infinitely-dimensional Fubini space with $b=1$.*

PROOF. By assumption we have relation (5) with

$$\Phi(z, \bar{z}) = \sum_{k=1}^{\infty} | f_k(z_\alpha) |^2.$$

On putting $\Phi = \log \psi$, that is, $\psi = e^\Phi$, we obtain

$$\psi = 1 + \sum_{k=1}^{\infty} | g_k(z_\alpha) |^2$$

for suitable functions $g_k(z_\alpha)$, and this is the substance of the theorem.

5. **Group invariance.** We consider a domain D in the space of the complex variables $z_1, \cdots, z_n$, and an arbitrary topological set Γ, and we consider a continuous function

$$f(\theta; z_\alpha) \tag{66}$$

in $\Gamma \times D$ which is analytic in D. We further assume that there exists some kind of integral in Γ such that

$$\Phi(z, \bar{z}) = \int_\Gamma f(\theta; z)\overline{f(\theta; z)}d\theta \tag{67}$$

exists in D, and that in the neighborhood of any point z^0 it is a uniform limit of an approximating sum

$$\sum_{r=1}^{s} \Delta_r f(\theta_r; z)\overline{f(\theta_r; z)}, \qquad \Delta_r > 0.$$

If we now form

$$g_{\alpha\beta^*} = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{68}$$

then our restriction (4) is met; also $g_{\alpha\beta^*}$ is non-negative, and we add the *explicit* assumption that it is strictly positive definite. The approximating tensor

$$h_{\alpha\beta^*} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \sum_{r=1}^{s} \left| \Delta_r^{1/2} f(\theta_r; z) \right|^2 \tag{69}$$

may be also assumed positive definite, and may be looked upon as resulting from an imbedding of D in Euclidean V_s. If we apply Theorem 12, and make the limiting passage from (69) to (68), we see that the resulting line-element (59) on D has nonpositive holomorphic and Ricci curvature.

We shall now vary the set-up slightly. We consider not a family of functions, but a family of analytic homeomorphisms $w_k = f_k(\theta; z_\alpha)$, $k = 1, \cdots, n$, of D into itself, and we assume that Γ is a locally compact group of such homeomorphisms. Correspondingly, we put

$$\Phi(z, \bar{z}) = \int_\Gamma \left(\sum_{k=1}^{n} \left| f_k(\theta; z) \right|^2 \right) d\theta \tag{70}$$

where $d\theta$ represents integration with respect to a Haar-measure, say a left-invariant one. Thus we have

$$g_{\alpha\beta^*}dz_\alpha d\bar{z}_\beta = \int_\Gamma \left(\sum_{k=1}^{n} \frac{\partial f_k(\theta, z)}{\partial z_\alpha} \overline{\frac{\partial f_k(\theta, z)}{\partial z_\beta}} \right) d\theta dz_\alpha d\bar{z}_\beta$$

(71)

$$= \int_\Gamma \left(\sum_{k=1}^{n} | dw_k(\theta, z) |^2 \right) d\theta.$$

Now, if θ' is a fixed element of Γ, and if we put $z'_\alpha = f_\alpha(\theta'; z) = \theta' z_\alpha$, then we have $w_k(\theta\theta'; z) = w_k(\theta; z')$, and we find that the line element is group invariant. We can now state the following very conditional result.

THEOREM 15. *If Γ is a transitive group of homeomorphisms of a domain D, if the general homeomorphism can be written in the form*

(72)
$$w_k = f_k(\lambda; a, \bar{a}; z_\alpha)$$

where $a = (a_1, \cdots, a_n)$ is an arbitrary point of D and λ is a set of parameters which describe the stability subgroup Γ_0 of Γ, if $d\lambda$ designates Haar-measure on Γ_0, if there is given in D a volume element

$$dv_a \equiv \rho(a, \bar{a}) \prod_{\alpha=1}^{n} da_\alpha d\bar{a}_\alpha$$

which is invariant under all transformations of Γ, and if the integrals

$$g_{\alpha\beta^*} = \int_{\Gamma_0 \times D} \left(\sum_{k=1}^{n} \frac{\partial f_k(\lambda; a, \bar{a}; z)}{\partial z_\alpha} \overline{\frac{\partial f_k(\lambda; a, \bar{a}; z)}{\partial z_\beta}} \right) dv_a d\lambda$$

converge uniformly in a neighborhood of every point z of D, then the corresponding Hermitian metric has no torsion and is group-invariant, and the curvature of holomorphic sections and the Ricci curvature are all nonpositive.

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