TOPOLOGICAL ABELIAN GROUPS WITH ORDERED NORMS

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The purpose of this paper is to study Abelian groups with a norm whose values are in another Abelian group having an order relation. Postulates on the ordering are given which are sufficient for the space to be a topological group under the neighborhood system of this norm. It is seen that the added assumption that the ordering is Archimedian implies the space is a subset of a normed linear space. A differential is defined in the general space. This is seen to be closely related to the Fréchet differential if the ordering is Archimedian.

DEFINITION 1. An *S-space* is a set S which is an Abelian group with a relation $\alpha > \beta$ (or $\alpha < \beta$) defined for some pairs α, β of elements of S and satisfying the postulates:

1. If $\alpha > \beta$, and $\beta > \gamma$, then $\alpha > \gamma$.
2. If $\alpha_1 > \alpha_2$ and $\alpha_3 \geqq \alpha_4$, then $\alpha_1 + \alpha_3 > \alpha_2 + \alpha_4$.
3. If $\alpha_1 > 0$, and $\alpha_2 > 0$, then there exists an $\alpha_3 > 0$ such that $\alpha_1 > \alpha_3$ and $\alpha_2 > \alpha_3$.
4. If $\alpha > 0$, then $\alpha \neq 0$.

DEFINITION 2. A set T with operations $x+y$ and $\|x\|$ defined for all elements x, y of T shall be said to be a G_s-space if the following are true:

1. T is an Abelian group with respect to $x+y$.
2. To every x in T and every positive number n there exists a positive integer N and an x_N such that $x = Nx_N$, where $N > n$.
3. $\|x\|$ is a function from T to an S-space with the following properties:

a. $\|x\| \geqq 0$, and $\|x\| = 0$ if and only if $x = 0$.
b. $\|x+y\| \leqq \|x\| + \|y\|$.
c. $\|nx\| = |n| \, \|x\|$ for all integers n.

By use of condition (2) of Definition 1, it is easily seen that an equivalent definition of S-spaces would result from replacing (4) by the condition that $\alpha > \beta$ implies the impossibility of $\alpha < \beta$. Also, the proof of Theorem 1 would be much simpler if it were assumed that $\alpha > 0$ whenever $n\alpha > 0$ for some positive integer n. Condition (2) of Definition 1 would then be unnecessary as far as this theorem is concerned. An S-space of a G_s-space is more restricted than the postulates of Definition 1 alone would indicate. This is shown by the following two theorems:

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THEOREM 1. *For any element* $\epsilon > 0$ *of an S-space, define the neighborhood* $U_{x_0,\epsilon}$ *of an element* x_0 *of an associated* G_s*-space as the totality of* $x \in G_s$ *satisfying* $n\|x-x_0\| < n\epsilon$ *for some positive integer n. Then* G_s *is a topological Abelian group if the neighborhood system is taken as the totality of all such neighborhoods.*

PROOF. (a) Clearly x_0 is in every $U_{x_0,\epsilon}$.

(b) Suppose U_{x_0,ϵ_1} and U_{x_0,ϵ_2} are two neighborhoods of x_0. Then $\epsilon_1 > 0$, $\epsilon_2 > 0$, and there is an $\epsilon_3 > 0$ such that $\epsilon_1 > \epsilon_2$ and $\epsilon_2 > \epsilon_3$. If $n\|x-x_0\| < n\epsilon_3$, then $n\|x-x_0\| < n\epsilon_1$ and $n\|x-x_0\| < n\epsilon_2$. Thus $U_{x_0,\epsilon_3} \subset U_{x_0,\epsilon_1} \cap U_{x_0,\epsilon_2}$.

(c) If y_0 is in $U_{x_0,\epsilon}$, then $p\|y_0-x_0\| < p\epsilon$ for some positive integer p. If $\epsilon' = p[\epsilon - \|y_0-x_0\|]$, then $U_{y_0,\epsilon'}$ contains some element $y \neq y_0$ if $U_{y_0,\epsilon'} \not\subset U_{x_0,\epsilon}$. Choose y_1 and $N \geqq p$ so that $y - y_0 = Ny_1$. Then there is a positive integer q such that $qN\|y_1\| < q\epsilon' = qp[\epsilon - \|y_0-x_0\|]$ and by (2) of Definition 1, $qp\|y_1\| < qp[\epsilon - \|y_0-x_0\|]$. Therefore $U_{y_0,\|y_1\|} \subset U_{x_0,\epsilon}$.

(d) If $x_0 \neq y_0$, then $\|x_0-y_0\| > 0$ follows from (3a) of Definition 2. There is then by (2) and (3c) of Definition 2 an element $u \in G_s$ and an integer $n \geqq 2$ such that $x_0 - y_0 = nu$ and $n\|u\| = \|x_0-y_0\|$. If $\epsilon = \|u\|$, then $U_{x_0,\epsilon} \cap U_{y_0,\epsilon} = 0$. For suppose z is in both of these neighborhoods. Then for some positive integer p, $pn\epsilon = p\|x_0-y_0\| \leqq p\|x_0-z\| + p\|z-y_0\| < 2\epsilon p$, or $pn\|u\| < 2p\|u\|$. Hence $p\|(n-2)u\| < 0$, which contradicts (3a) of Definition 2.

This verifies that any G_s-space is a Hausdorff space [1, pp. 228–229, (A), (B), (C), (5)].[1] To show that $x+y$ is continuous, let $U_{x+y,\epsilon}$ be any neighborhood of $x+y$. If there is an element $z \neq x+y$ in $U_{x+y,\epsilon}$, then choose u with $z-(x+y) = nu$ and $n > 2$. As in (d) above, it follows that if $\epsilon_1 = \|u\|$, then $x'+y' \in U_{x+y,\epsilon}$ if $x' \in U_{x,\epsilon_1}$ and $y' \in U_{y,\epsilon_1}$. If the only element in $U_{x+y,\epsilon}$ is $x+y$, then $U_{x,\epsilon}$ contains only x and $U_{y,\epsilon}$ contains only y. For if $p\|z-x\| < \epsilon p$, then $p\|z+y-(x+y)\| < \epsilon p$ and $z+y \in U_{x+y,\epsilon}$; while if $q\|z-y\| < \epsilon q$, then $z+x \in U_{x+y,\epsilon}$. Thus $x+y$ is continuous. Since $y \in U_{x,\epsilon}$ implies $-y \in U_{-x,\epsilon}$, $-x$ is also continuous.

DEFINITION 3. An S-space is *Archimedean* if for every $\epsilon_1 > 0$ and $\epsilon_2 > 0$ of S there exists a positive integer n such that $\epsilon_2 < n\epsilon_1$.

The following is an example of a G_s-space whose S-space is not Archimedean. Let S be the set of complex numbers, and G_s the same set. Let $\|x+yi\| = |x| + |y|i$, and $a+bi > c+di$ if and only if $b > d$ or $b = d$ and $a > c$. These spaces clearly satisfy the postulates of Defi-

[1] Numbers in brackets refer to the references cited at the end of the paper.

nitions 1 and 2. But if $\alpha=1+0\cdot i$ and $\beta=1+i$, then there is no number n such that $n\alpha>\beta$.

Let S be the set of all real numbers, G_s the set of all complex numbers $x+yi$ with x and y rational, and $\|x+yi\|=|x+yi|$. This is a G_s-space whose S-space is Archimedian, but which is itself a subset of the normed linear space of all complex numbers. The meaning of assuming that an S-space is Archimedian is further shown by the following results.

LEMMA 1. *A G_s-space whose S-space is Archimedian is a normable topological Abelian group.*

PROOF. Let G_1 be a G_s-space whose S-space S_1 is Archimedian. Let U_ϵ be any neighborhood of zero. Then G_1 is normable if U_ϵ generates G_1 and U_ϵ is bounded and convex [**2**, Corollary 5.2].

(a) *U_ϵ generates G_1.* For let $x\in G_1$ and choose an integer n such that $n\epsilon>\|x\|$. There is then a number $N>n$ and an x_N such that $x=Nx_N$. But then $N\epsilon>\|x\|=N\|x_N\|$ and $N\|x_N\|<N\epsilon$. Thus $x_N\in U_\epsilon$ and $x\in U_\epsilon^N=U_\epsilon+\cdots U_\epsilon$.

(b) *U_ϵ is bounded* [**2**, Definition 2.3]. For let $U_{\epsilon'}$ be any other neighborhood of the identity and choose n so that $n\epsilon'>\epsilon$. Suppose that $mx\in U_\epsilon$ for an $m\geqq n$. Then $p\|mx\|<p\epsilon<pn\epsilon'\leqq pm\epsilon'$ for some positive integer p. Hence $pm\|x\|<pm\epsilon'$ and $x\in U_{\epsilon'}$.

(c) *U_ϵ is convex* [**2**, Definition 2.1]. For suppose that $nx\in U_\epsilon^n$ for a positive integer n. Then $nx=x_1+x_2+\cdots+x_n$, where for each x_i there is a positive integer p_i such that $p_i\|x_i\|<p_i\epsilon$. Thus $n(p_1p_2\cdots p_n)\|x\|<m(p_1p_2\cdots p_n)\epsilon$ and $x\in U_\epsilon$.

It follows from Lemma 1 that if S is Archimedian, then G_s is a normable topological Abelian group and therefore a subspace of a normed linear space [**2**, Theorem 4.1]. The following is a similar, but stronger, result. Hereafter a G_s-space whose S-space is Archimedian will be called an *Archimedian G_s-space.*

THEOREM 2. *For a given Archimedian G_s-space G_1 there is a normed linear space T_1 which contains G_1 as a subset and which is contained in every normed linear space containing G_1. Also, every element of T_1 is a limit point of elements of G_1.*

PROOF. As noted above, there is a normed linear space T with $G_1\subset T$. Then $\sum_{i=1}^n a_ix_i\in T$ for any numbers a_i and elements x_i of G_1, such multiplication being defined in T. The set T_1 of all such elements of T is clearly a linear space and as a linear subset of T is a normed linear space. Because of (2) of Definition 2, for any number a and element $x_0 \in G_1$ there are numbers α_i such that $\lim_{i\to a}\alpha_i = a$ and $\alpha_i x_0$ is an element of G_1. Let $x = \sum_{i=1}^n a_i x_i \in T_1$, where $x_i \in G_1$, and let ϵ be any positive number. Choose numbers α_i such that $|a_i - \alpha_i| < \epsilon/n\|x_i\|$ and $\alpha_i x_i \in G_1$ for each i. Then $\|x - \sum_{i=1}^n \alpha_i x_i\| < \epsilon$ and $\sum_{i=1}^n \alpha_i x_i \in G_1$. Hence every element of T_1 is a limit point of elements of G_1. From this, it is clear that $T_1 \subset T'$ for any normed linear space T' containing G_1.

The usual definition of a linear function as being additive and continuous will be used in the following. If the G_s-space is Archimedian, then the theory of the differential as defined below becomes a consequence of the established theory of Fréchet differentiation of functions with arguments and values in normed linear spaces.

DEFINITION 4. Let G_1 and G_2 be any two G_s-spaces. If $f(x)$ is a function on a neighborhood U_1 of $x_0 \in G_1$ to G_2, then $f(x)$ is differentiable at $x = x_0$ if there exists a function $f(x_0; \delta x)$ on G_1 to G_2, defined for all elements δx of G_1 and such that:

1. $f(x_0; \delta x)$ is linear in δx.
2. For any integer $n > 0$ there exists a $\rho > 0$ such that $n\|f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x)\| < \|\delta x\|$ for all δx such that $0 < \|\delta x\| < \rho$.

In this case, $f(x_0; \delta x)$ is called the differential of $f(x)$ at x_0.

THEOREM 3. *Let G_1 and G_2 be any two Archimedian G_s-spaces. If the function $f(x)$ on $U_1 \subset G_1$ to G_2 has the differential $f(x_0; \delta x)$ at $x_0 \in U_1$, and T_1 and T_2 are the smallest normed linear spaces containing G_1 and G_2, respectively, then there is a function F on T_1 to T_2 such that:*

1. *$F(x) = f(x)$ for all $x \in U_1$.*
2. *F has the differential $F(x_0; \delta x)$ at x_0, where $F(x_0; \delta x)$ is the unique linear function on T_1 to T_2 for which $F(x_0; \delta x) = f(x_0; \delta x)$ for all $x \in G_1$.*

PROOF. Let x be any element of T_1. The existence of elements $x_i \in G_1$ with $x = \lim_{i\to\infty} x_i$ follows from Theorem 2. Define $F(x_0; x)$ as $\lim_{i\to\infty} f(x_0; x_i)$. Such sequences are clearly Cauchy sequences and converge to an element of any Banach space containing T_2. But $F(x_0; x)$ is then a linear function on T_1 to this Banach space, and is therefore homogeneous of degree one in x. Since $x = \sum_{i=1}^n a_i y_i$ for some elements $y_i \in G_1$ and numbers a_i, this implies that $F(x_0; \sum_{i=1}^n a_i y_i) = \sum_{i=1}^n a_i F(x_0; y_i)$ and is an element of T_2 $[F(x_0; y_i) = f(x_0; y_i) \in G_2$ if $y_i \in G_1]$. Thus $F(x_0; x)$ is a linear function on T_1 to T_2. Since G_1 is dense in T_1, it follows that this is the only linear function with $F(x_0; \delta x) = f(x_0; \delta x)$ for $\delta x \in G_1$. Now define F on T_1 by the relation $F(x) = f(x)$ if $x \in U_1$, and

$$F(x) = F(x_0) + F(x_0; x - x_0)$$

otherwise. If n is any positive integer, then there is a number $\rho>0$ such that $0<\|\delta x\|<\rho$ implies $x_0+\delta x\in U_1$ and $n\|f(x_0+\delta x)-f(x_0)-f(x_0;\delta x)\|<\|\delta x\|$ if $\delta x\in G_1$. But then $n\|F(x_0+\delta x)-F(x_0)-F(x_0;\delta x)\|<\|\delta x\|$ if $0<\|\delta x\|<\rho$ and $\delta x\in T_1$, since this is satisfied for δx if $\delta x\in G_1$ and $0<\|\delta x\|<\rho$, while $F(x_0+\delta x)=F(x_0)+F(x_0;\delta x)$ if δx is not in $U_1\subset G_1$. Hence F has the differential $F(x_0;\delta x)$ at x_0; while $F(x)=f(x)$ if $x\in U_1$ and $F(x_0;x)=f(x_0;x)$ if $x\in G_1$.

THEOREM 4. *Let T_1 and T_2 be any two normed linear spaces. If a function f on T_1 to T_2 has a differential $f(x_0;\delta x)$ at the point x_0 (in the sense of Definition 4), then $f(x_0;\delta x)$ is a Fréchet differential of f at x_0.*

PROOF. Let ϵ be any positive number and choose an integer n such that $0<1/n\leqq\epsilon$. By assumption, there is a number ρ such that $n\|f(x_0+\delta x)-f(x_0)-f(x_0;\delta x)\|<\|\delta x\|$ for all δx with $0<\|\delta x\|<\rho$. But then $\|f(x_0+\delta x)-f(x_0)-f(x_0;\ \delta x)\|<\epsilon\|\delta x\|$ if $0<\|\delta x\|<\rho$. Hence $f(x_0;\delta x)$ is the Fréchet differential of $f(x)$ at x_0.

COROLLARY 1. *Let G_1 and G_2 be any two Archimedian G_s-spaces, and $f(x)$ be a function on a neighborhood of $x_0\in G_1$ to G_2. Then*: (1) *If a differential of $f(x)$ exists, it is unique.* (2) *If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$.* (3) *If $f(x)$ is differentiable at $x=x_0$, then it is also differentiable in the sense of Michal* [3], *and the two differentials are equal.*

COROLLARY 2. *Let G_1, G_2, and G_3 be three Archimedian G_s-spaces. Suppose $\phi(x)$ is a function on a neighborhood of $x_0\in G_1$ to G_2 and is differentiable at x_0, while $f(\phi)$ is a function on a neighborhood of $\phi(x_0)\in G_2$ to G_3 which is differentiable at $\phi(x_0)$. Then $f[\phi(x)]$ is differentiable at x_0, and this differential is $f[\phi(x_0);\phi(x_0;\delta x)]$.*

REFERENCES

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