without nilpotent elements is a field (cf. [1, Lemma 2]), but we shall not do this here.

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A NOTE ON RELATIVELY PRIME SEQUENCES

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In volume 2 of Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, pp. 133 and 342, there occurs the following result (appearing also in Hardy-Wright, *Theory of numbers*, p. 14):

Theorem 1. *No two numbers of the form* $2^{2^n}+1$, $n=1, 2, \cdots$, *have a common divisor greater than* 1.

The numbers $2^{2^n}+1$, $n=1, 2, \cdots$, are the well known Fermat numbers, which may be generated by iteration of the quadratic polynomial $\phi(x)=(x-1)^2+1$, choosing x equal to 3. This follows easily by induction, since, putting $\phi_1(x)=\phi(x)$, $\phi_{n+1}(x)=\phi(\phi_n(x))$, if $\phi_n(x)=2^{2^n}+1$, then $\phi_{n+1}(x)=2^{2^{n+1}}+1$.

The above observation leads to the following result of which Theorem 1 is a special case:

Theorem 2. *Let* $\phi(x)$ *be a polynomial in* x *with integral coefficients*

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possessing the following properties:

$$\phi_n(0) = \phi(0), \qquad n \geqq 1, \qquad \phi(0) \neq 0, \tag{1}$$

$$(x, \phi(0)) = 1 \rightarrow (\phi(x), \phi(0)) = 1. \tag{2}$$

Then if x is an integer and $(x, \phi(0))=1$, no two of the numbers $x, \phi_1(x), \cdots, \phi_n(x), \cdots$, have a common divisor greater than 1.

PROOF. Let us assume that the theorem is false, so that for some $m>1$, $n>m$, we have $(\phi_n(x), \phi_m(x))>1$. Since $\phi_n(x)=\phi_{n-m}(\phi_m(x)) \equiv \phi_{n-m}(0) \bmod \phi_m(x) \equiv \phi(0) \bmod \phi_m(x)$, if $(\phi_m(x), \phi_n(x))>1$, $(\phi_n(x), \phi(0))>1$. However, since $(x, \phi(0))=1$, it follows that $(\phi(x), \phi(0))=1$, and thus that $(\phi_n(x), \phi(0))=1$, which is a contradiction.

The sequence $x, \phi_1(x), \cdots, \phi_n(x), \cdots$, will have an infinity of distinct prime divisors if there are an infinite number of terms of the sequence different from ± 1. This is true if x is an integer such that for $y \geqq x$, $\phi(y)>y$,[1] or if x is such that $\phi(x)>x$ and x is greater than the roots of $\phi(x)=\pm 1$, or, finally, if $|\phi(\pm 1)|>1$.

It is easy to verify that $\phi(x)=(x-1)^2+1$ satisfies the conditions of Theorem 2. If we choose $x=3$, we obtain the Fermat numbers, as mentioned above. Another admissible polynomial is $(x-2)^4-12$, where we shall choose x satisfying the conditions $(x, 4)=1$, $x \geqq 5$.

Theorem 2 leads one to consider the following question:[2]

Consider an irreducible polynomial $f(x)$ with integral coefficients, and choose an integer x so that all the iterates $f_n(x)$ yield distinct numbers. Can all these numbers be primes?

That this question is probably very difficult to answer might be surmised from the fact that the primality of *all* the Fermat numbers was disproved by exhibiting a specific counter-example, and the behavior of the general term of the sequence $2^{2^n}+1$ is still undetermined.

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[1] These latter alternatives were suggested by the referee, who also pointed out some superfluous restrictions in the original statement of Theorem 2.

[2] The case where $f(x)$ is linear has been worked out by the author and H. N. Shapiro, and the answer is negative.