# A NOTE ON TRANSFORMS OF UNBOUNDED SEQUENCES

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During an evening session at the recent Ithaca meeting it was conjectured that it is possible to construct a regular Toeplitz matrix $A\equiv\|a_{nk}\|$ with the property that for every sequence $s_n$ the transformed sequence

$$t_n = \sum_{k=0}^{\infty} a_{nk}s_k \tag{1}$$

possesses at least one limit point in the finite plane; and it was counter-conjectured that for every regular Toeplitz matrix $A$ there exists a sequence $s_n$ such that the sequence $t_n$ of equation (1) tends to infinity monotonically. It is the purpose of the present note to report that both conjectures are false and to prove a consolation theorem regarding the first conjecture. The notation of equation (1) will be used throughout the paper.

THEOREM 1. *If $A$ is a row-finite regular Toeplitz matrix, there exist sequences $s_n$ such that the corresponding sequences $|t_n|$ tend to infinity with arbitrary rapidity.*

Let $A$ be a row-finite regular Toeplitz matrix. If $n_0$ is sufficiently large, each row whose index exceeds $n_0$ contains a nonzero element, and therefore a last nonzero element (a *row-terminal* element). Let $k_1, k_2, \cdots$ be the indices of the columns that contain row-terminal elements ($k_1<k_2<\cdots$), and let the terms $s_k$ ($k\neq k_1, k_2, \cdots$) be chosen arbitrarily. Regularity of the matrix $A$ implies that each column contains at most a finite number of row-terminal elements, that is, that for each column the row-terminal elements are bounded away from zero. It is now clear that if $f(n)$ is any arbitrary real function, the terms $s_{k_1}, s_{k_2}, \cdots$ can be chosen large enough so that $|t_n|>f(n)$ $(n>n_0)$, and Theorem 1 is proved.

THEOREM 2. *If $A$ is a regular Toeplitz matrix, there exists a sequence $s_n$ such that the sequence $t_n$ has no limit point in the finite plane.*

Here the matrix $A$ is not required to be row-finite, and if the sequence $s_n$ is chosen (as in the proof of Theorem 1) so as to tend to infinity with reckless rapidity, there is danger that the transformation $A$ does not apply to the sequence (*the transformation $A$ applies to the*

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*sequence $s_n$ if the sum $\sum_{k=0}^{\infty} a_{n_k s_k}$ exists in the ordinary sense when $n$ is sufficiently large*). Let $c$ be a constant such that $\sum_{k=0}^{\infty}|a_{nk}| < c/5$ for all $n$. (Note: this implies that $c \geqq 5$.) Integers $m_1, n_1, m_2, n_2, \cdots, m_r, n_r, \cdots$ can be chosen successively so that

$$\sum_{k=m_1+1}^{\infty} |a_{0k}| < 1/c,$$

$$\sum_{k=0}^{m_1} |a_{nk}| < 1/5 \qquad \text{when } n > n_1,$$

$$\sum_{k=m_2+1}^{\infty} |a_{nk}| < 1/c^2 \qquad \text{when } n \leqq n_1,$$

$$\sum_{k=0}^{m_2} |a_{nk}| < 1/5 \qquad \text{when } n > n_2,$$

and so that generally

$$\sum_{k=m_r+1}^{\infty} |a_{nk}| < 1/c^r \qquad \text{when } n \leqq n_{r-1} \ (r = 1, 2, \cdots; n_0 = 0),$$

$$\sum_{k=0}^{m_r} |a_{nk}| < 1/5 \qquad \text{when } n > n_r \ (r = 1, 2, \cdots).$$

The sequence $s_n$ is now chosen according to the rule

$$s_k = (1 + 1/c)^r \quad \text{when } m_{r-1} < k \leqq m_r \ (r = 1, 2, \cdots; m_0 = -1).$$

To see that the transform of this sequence tends to infinity, observe that when $n_{r-1} < n \leqq n_r$

$$\text{(i)} \quad \sum_{0}^{m_{r-1}} |a_{nk}| s_k < \frac{1}{5}(1 + 1/c)^{r-1} \qquad (r = 2, 3, \cdots);$$

$$\text{(ii)} \quad \begin{aligned} \sum_{m_{r+1}+1}^{\infty} |a_{nk}| s_k &< \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} + \frac{1}{c^{r+2}}(1 + 1/c)^{r+3} + \cdots \\ &= \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} \frac{1}{1 - (1/c)(1 + 1/c)} \\ &< \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} \frac{1}{1 - 2/5} \\ &\leqq \frac{1}{3c^r}(1 + 1/c)^{r+2} \qquad (r = 0, 1, \cdots); \end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned} \sum_{m_{r-1}+1}^{m_r+1}{}' a_{nk}s_k &= (1 + 1/c)^r[\alpha_n + \beta_n(1 + 1/c)] \\ &= (1 + 1/c)^r[\alpha_n + \beta_n + \beta_n/c] \quad (r = 1, 2, \cdots) \end{aligned}$$

where

$$\alpha_n = \sum_{m_{r-1}+1}^{m_r} a_{nk}, \qquad \beta_n = \sum_{m_r+1}^{m_r+1} a_{nk}, \qquad |\beta_n|/c < 1/5,$$

and, when $n$ is sufficiently large,

$$|\alpha_n + \beta_n| > 3/5.$$

It follows that for all sufficiently large values of $n$

$$\begin{aligned} \left| \sum_0^\infty a_{nk}s_k \right| &> (1 + 1/c)^r \left[ \frac{2}{5} - \frac{1}{5(1 + 1/c)} - \frac{1 + 1/c}{3c^r} \right] \\ &> \frac{1}{10}(1 + 1/c)^r, \end{aligned}$$

and the proof is complete.

Only slight modifications in the last construction are necessary for the following result:

THEOREM 2a. *If $A$ is a regular Toeplitz matrix, there exists a sequence $s_n$ such that $t_n = \rho_n e^{i\theta_n}$, with $\lim_{n\to\infty}\rho_n = \infty$ and $\lim_{n\to\infty}\theta_n = 0$. If the matrix $A$ is also real, the sequence $s_n$ can be chosen so that the sequence $t_n$ is real and positive.*

In the inequality governing the choice of $n_r$, the constant $1/5$ must be replaced by a quantity that tends to zero as $r$ becomes large. Also, the growth of $s_n$ must be so slow that the ratio between the coefficients of $\alpha_n$ and $\beta_n$ respectively tends to unity as $r$ becomes large (this can be achieved by choosing $s_k = (1+1/c)^{r^{1/2}}$ when $m_{r-1} < k \leq m_r$). Since $\lim_{n\to\infty}(\alpha_n + \beta_n) = 1$ and $\beta_n$ is bounded, the theorem then follows immediately.

In every regular Toeplitz matrix, the early and the late elements in each row are small. Roughly speaking, the rate of growth of the sequence constructed above (and therefore the rate of growth of the transformed sequence) is retarded by the slowness with which the early and the late elements tend to zero. The proof of the following theorem indicates that this retarding effect is not a mere accident of the construction; the theorem itself salvages a portion of the conjecture that gave occasion to this note.

THEOREM 3. *If $f(n)$ is a real function such that $\lim_{n\to\infty} f(n) = \infty$, there exists a regular Toeplitz matrix $A$ such that, for every sequence $s_n$ to which the transformation $A$ is applicable, the inequality*

$$|t_n| < f(n) \tag{2}$$

*is satisfied for infinitely many values of $n$.*

The theorem asserts that there exist regular transformations that transform every sequence to which they are applicable either into a sequence with at least one finite limit point, or else into a sequence whose terms tend to infinity (in absolute value) with arbitrary slowness independent of the sequence. The following matrix satisfies the promise of the theorem: If $n$ is even, $a_{nk}=1$ when $k=n/2$, and otherwise $a_{nk}=0$. If $n$ is odd, $a_{nk}=1$ when $k=(n-1)/2$; $a_{nk}=0$ when $k<(n-1)/2$; when $k>(n-1)/2$, $a_{nk}$ takes the value zero except when $k$ has one of the values $k_1, k_2, \cdots$ about to be described. The number $k_r$ $(r=1, 2, \cdots)$ is chosen so that $f(k_r)>2^r$; and if $n$ is odd and $k_r>(n-1)/2$, $a_{nk_r}=2^{-r}$.

Suppose now that $s_n$ is a sequence such that the inequality (2) does not hold infinitely often. Because of those rows of $A$ that contain only one nonzero element, it is necessary that $|s_k| \geqq f(k)$ for all sufficiently large $k$. This implies that for odd values of $n$ the formal series $\sum_{k=0}^{\infty} a_{nk}s_k$ contains infinitely many terms whose absolute value exceeds unity, and the transformation $A$ cannot be applicable to the sequence $s_n$.

THEOREM 4. *There exists a regular Toeplitz matrix $A$ such that the sequence $|t_n|$ does not increase monotonically for any sequence $s_n$.*

Example: If $n$ is even, choose $a_{nk}=1$ when $k=n/2$, otherwise $a_{nk}=0$. If $n$ is odd, choose $a_{nk}=1-1/n$ when $k=(n-1)/2$, otherwise $a_{nk}=0$.

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