# SOME INEQUALITIES RELATING TO CONFORMAL MAPPING UPON CANONICAL SLIT-DOMAINS

BERNARD EPSTEIN

Let $D$ be a domain of the extended $z$-plane ($z=x+iy$) of finite connectivity $n$, which contains the point $z=\infty$ and is bounded by $n$ proper[1] continua. According to a fundamental theorem in the theory of conformal mapping of multiply-connected domains [4, 7][2] there exists one and only one function $\zeta=s_\theta(z)$ which in the neighborhood of $z=\infty$ has a Laurent expansion of the form

$$\zeta = s_\theta(z) = z + \frac{a_\theta}{z} + \cdots \tag{1}$$

and which maps $D$ conformally and bi-uniformly upon a domain $D_\theta$ of the $\zeta$-plane bounded by $n$ rectilinear slits each of which makes the angle $\theta$ with the positive direction of the real axis. The domain $D_\theta$ is itself also uniquely determined for each value of $\theta$.

In the present paper we shall derive two inequalities involving the coefficient $a_\theta$ appearing in (1) and the outer measure $A$ of the complement (with respect to the entire plane) of the domain $D$—that is, the greatest lower bound of the total area enclosed by a set of analytic curves surrounding the boundary continua. The first of these inequalities is the following:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) \geqq \frac{A}{2\pi}\cdot \tag{2}$$

The second inequality, which will be derived by using the theory of orthonormal systems of analytic functions [1, 2, 9, 10], constitutes a strengthening of (2), namely:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) - \frac{|a_\theta|^2}{a_0 - a_{\pi/2}} \geqq \frac{A}{2\pi}\cdot \tag{3}$$

It suffices to prove the inequalities (2) and (3) for the case when the boundary continua of $D$ are closed analytic curves $C_1, C_2, \cdots, C_n$, for it is known that $D$ can be approximated by an increasing sequence of domains having such boundaries for which the mapping functions corresponding to (1) will converge to $s_\theta(z)$, so that (2) and

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[1] A proper continuum is one which does not consist of a single point.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

(3) will continue to hold in the limit, when $A$ is interpreted in the manner explained above.

Since the boundary of $D$ is for the present assumed to consist of analytic curves, it follows that $s_\theta(z)$ remains analytic there as well as in the interior of $D$. It then follows, taking account of the form of $s_\theta(z)$ in the neighborhood of $z=\infty$, that $d(s_\theta-z)/dz$ is quadratically integrable over $D$, that is,

$$I_\theta = \iint_D \left| \frac{d}{dz}(s_\theta - z)\right|^2 dxdy \tag{4}$$

exists. $I_\theta$ is real and non-negative, vanishing if and only if $s_\theta=z$, that is, if and only if $D$ is identical with the domain into which it is to be mapped. Now the double integral appearing in (4) can be transformed into an integral taken along the boundary curves of $D$, as follows:

$$I_\theta = -\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \overline{(s_\theta - z)}\left(\frac{ds_\theta}{dz} - 1\right)dz, \tag{5}$$

the sense of integration being positive with respect to the *interior* of each boundary curve. By multiplying out the integrand, $I_\theta$ can be expressed as the sum of four terms, namely:

$$-\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{s}_\theta \frac{ds_\theta}{dz}dz, \tag{6a}$$

$$\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{s}_\theta dz, \tag{6b}$$

$$\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{z}\frac{ds_\theta}{dz}dz, \tag{6c}$$

$$-\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{z}dz. \tag{6d}$$

The integrals (6a), (6b), (6c) will now be evaluated by employing an artifice which has been used with great success by Grunsky [4]. One observes that on each curve $C_k$ the expression $e^{-i\theta}(s_\theta-\zeta_k)$ is real, where $\zeta_k$ is any point on or collinear with the image of $C_k$, so that

$$\bar{s}_\theta = e^{-2i\theta}s_\theta + \text{constant}.^3 \tag{7}$$

Replacing $\bar{s}_\theta$ in (6a) by the right-hand side of (7) and recalling that $s_\theta(z)$ is single-valued in $D$, one sees that (6a) vanishes. By performing the same substitution in (6b) and then integrating around a

[3] In general, of course, this constant will be different for each $C_k$.

large circle lying entirely within the domain of convergence of the expansion (1) instead of around the curves $C_k$ (which is justified by the Cauchy integral theorem), one finds that (6b) is equal to $\pi a_\theta e^{-2i\theta}$.

(6c) may be evaluated most easily by integration by parts; the integrated part vanishes, leaving an integral which is the conjugate of (6b). Thus, (6c) is equal to $\pi \bar{a}_\theta e^{2i\theta}$.

Finally, (6d) is easily seen to be equal to $-A$. Thus, equation (4) may be rewritten in the form

$$I_\theta = -A + 2\pi \operatorname{Re}(a_\theta e^{-2i\theta}). \tag{8}$$

Since $I_\theta$ is of positive definite character, vanishing, as mentioned above, if and only if $s_\theta = z$, the inequality (2) follows directly from (8).[4] Since the real part of a complex number is at most equal to its absolute value, there is immediately obtained from (2) the following lower bound on $|a_\theta|$, valid for all values of $\theta$:

$$|a_\theta| \geqq \frac{A}{2\pi}\cdot \tag{9}$$

That the factor $1/2\pi$ appearing in (2) cannot be replaced by any larger number is easily seen by considering the mapping of the exterior of an ellipse upon the exterior of a slit parallel to the major axis and letting the eccentricity of the ellipse approach unity. For example, for $\theta=0$, the exterior of the ellipse $x^2R^4/(R^2+1)^2+y^2R^4/(R^2-1)^2=1$ $(R>1)$ is mapped upon the exterior of the slit $[-2, 2]$ of the $\zeta$-plane by eliminating $w$ between the pair of equations:

$$z = w + 1/R^2w, \tag{10a}$$ $$\zeta = w + 1/w. \tag{10b}$$

It is easily found that the Laurent expansion for $\zeta$ at $z=\infty$ is of the form

$$\zeta = z + \frac{(1-1/R^2)}{z} + \cdots \tag{11}$$

so that, in this case:

$$\frac{\operatorname{Re}(a_\theta e^{-2i\theta})}{A} = \frac{1-1/R^2}{\pi(1-1/R^4)} = \frac{R^2}{\pi(1+R^2)}\cdot \tag{12}$$

Letting $R$ (and hence the eccentricity) approach unity, the right-hand side of (12) is seen to approach $1/2\pi$.

[4] The evaluation of positive-definite integrals has been used frequently as a means of obtaining inequalities in the theory of conformal mapping. To mention only one example, one may cite the "Flächensatz" of Bieberbach [3].

A more refined estimate than (9) is easily obtained in the following manner. First, by setting $\theta=0$, $\theta=\pi/2$, $\theta=\pi/4$, and $\theta=3\pi/4$, respectively, there are obtained from (2) the inequalities:

(13a) $\operatorname{Re} a_0 \geqq \dfrac{A}{2\pi}$, (13c) $\operatorname{Im} a_{\pi/4} \geqq \dfrac{A}{2\pi}$,

(13b) $\operatorname{Re} a_{\pi/2} \leqq -\dfrac{A}{2\pi}$, (13d) $\operatorname{Im} a_{3\pi/4} \leqq -\dfrac{A}{2\pi}\cdot$

(Grunsky, in [5], and Schiffer in [8] and [10], showed not only that $a_0-a_{\pi/2}$ (the "span" of $D$) is real, but that

$$a_0 - a_{\pi/2} \geqq \frac{2A}{\pi}\cdot \tag{14}$$

While neither of these results can be obtained from the considerations presented here, it is of interest that previously no estimates like (13a) and (13b) for these coefficients separately had been obtained.)

Now it is known [4] that for each value of $\theta$, $s_\theta(z)$ can be expressed as a simple combination of $s_0(z)$ and $s_{\pi/2}(z)$, namely

$$s_\theta(z) = e^{i\theta}\{s_0 \cos\theta - is_{\pi/2}\sin\theta\}. \tag{15}$$

Combining (15) with (1), the following equation is obtained for $a_\theta$:

$$a_\theta = e^{i\theta}\{a_0\cos\theta - ia_{\pi/2}\sin\theta\}. \tag{16}$$

Now, if we set $a_0=\alpha+i\beta$ and $a_{\pi/2}=\gamma+i\beta$ (here we use the fact that $a_0-a_{\pi/2}$ is real), it is found that (1) may be written in the form

$$\frac{\alpha-\gamma}{2} + \frac{\alpha+\gamma}{2}\cos 2\theta + \beta\sin 2\theta \geqq \frac{A}{2\pi}\cdot \tag{17}$$

Since this inequality holds for all values of $\theta$, it holds when $\theta$ is so chosen as to minimize the left-hand side of (17). The result obtained is

$$\frac{\alpha-\gamma}{2} - \left(\left(\frac{\alpha+\gamma}{2}\right)^2 + \beta^2\right)^{1/2} \geqq \frac{A}{2\pi}, \tag{18}$$

which is completely equivalent to (17).

The inequality (18) admits an interesting geometrical interpretation, which may be seen in the following manner. From (16) and the real character of $a_0-a_{\pi/2}$, it is easily shown that the point $a_\theta$, when plotted in the usual manner, describes the circumference of a circle with radius $R=(\alpha-\gamma)/2$ and with center at the point $(\alpha+\gamma)/2+i\beta$, whose distance $r$ from the origin is precisely the second term of the left-hand side of (18). Thus, it is seen that the left-hand side of (18) is, both in magnitude and sign, equal to the distance between the origin and the nearest point of the circle described by the coefficients $a_\theta$. The inequality (9) may be seen geometrically from (18), while it is clear from either (13a–d) or (18) that the origin lies inside (or perhaps on) the aforementioned circle. This latter fact has previously been obtained in [4] by Grötzsch, who showed that for the class of all biuniform conformal mappings of a fixed domain $D$ (of the type described at the beginning of this paper) having at infinity a Laurent expansion of the form

$$\zeta = z + \frac{c}{z} + \cdots, \tag{19}$$

the coefficients $c$ cover a circular area of the complex plane whose circumference is described by the points $a_\theta$; since $\zeta = z$ is a mapping belonging to the aforementioned class, it follows that the origin cannot be exterior to the circle. By combining this observation with the fact that $a_0 - a_{\pi/2}$ is real and equal to the diameter of the circle described by the coefficients $a_\theta$, the inequalities

(20a) $\operatorname{Re} a_0 \geqq 0$, (20b) $\operatorname{Re} a_{\pi/2} \leqq 0$

are obtained. These results are contained in (13a) and (13b), which were obtained by much simpler considerations than those used by Grötzsch.

As stated previously, we can strengthen the inequality (2) by the use of the theory of complete systems of orthonormal functions. As shown in [1], there can be chosen for every domain $B$ whose boundary does not consist entirely of isolated points—that is, contains at least one proper continuum—in infinitely many ways an orthonormal[5] sequence of functions $\{f_k(z)\}$, each of which is analytic and uniform, and possesses a uniform integral in $B$, such that every function $f(z)$ defined in $B$ which satisfies the above conditions and is quadratically integrable over the entire domain can be expanded in a series of the form

$$f(z) = \sum_{k=1}^{k=\infty} c_k f_k(z) \tag{21}$$

which converges uniformly to $f(z)$ in every closed subdomain of $B$.

[5] Orthonormality is defined by the usual condition: $\iint_B f_m(z)\overline{f_n(z)}\,dxdy = \delta_{mn}$.

Such a sequence of functions is called a complete system for the domain $B$.

It is known that the kernel function of the domain, defined as follows:

$$K(z, \bar{z}_1) = \sum_{k=1}^{k=\infty} f_k(z) \overline{f_k(z_1)}, \tag{22}$$

is independent of the particular choice of the complete orthonormal system $\{f_k(z)\}$. Identifying $B$ with the domain $D$ considered in §2, let two different complete orthonormal systems be constructed, beginning with the functions:[6]

$$g(z) = \frac{ds_\theta/dz - 1}{(2\pi \operatorname{Re}(a_\theta e^{-2i\theta}) - A)^{1/2}} \tag{23a}$$

and

$$h(z) = \frac{d(s_0 - s_{\pi/2})/dz}{(2\pi(a_0 - a_{\pi/2}))^{1/2}} \tag{23b}$$

respectively. (It follows from (8) that $g(z)$ is normalized, while Schiffer [10] has shown this for $h(z)$.)

Letting $z_1 = z$, it is obvious from (22) that

$$K(z, \bar{z}) \geqq | g(z) |^2. \tag{24}$$

Multiplying both sides of (24) by $|z|^4$ and letting $|z| \to \infty$, one obtains, taking account of (1) and (2),

$$\lim_{|z|\to\infty} | z |^4 K(z, \bar{z}) \geqq \frac{| a_\theta |^2}{2\pi \operatorname{Re}(a_\theta e^{-2i\theta}) - A} \geqq 0. \tag{25}$$

On the other hand, it is proven in [10] that every function which belongs to the class described at the beginning of this section and which is orthogonal to $h(z)$ vanishes at $z = \infty$ to at least the third order, so that

$$\lim_{|z|\to\infty} | z |^4 K(z, \bar{z}) = \lim_{|z|\to\infty} | z |^4 | h(z) |^2 = \frac{1}{2\pi}(a_0 - a_{\pi/2}). \tag{26}$$

Combining (25) with (26) one obtains the inequality (3), which is seen to constitute a strengthening of (2). Replacing $a_\theta$ in (3) by the right-hand side of (16), one finds that the left-hand side of (3) is inde-

[6] Here we disregard the special case $s_\theta = z$, for which (3) is directly seen to hold.

pendent of $\theta$, so that (3) may be rewritten in the simpler form

$$-\frac{\alpha\gamma+\beta^2}{\alpha-\gamma} \geqq \frac{A}{2\pi}. \tag{27}$$

(Here, as before, $a_0=\alpha+i\beta$, $a_{\pi/2}=\gamma+i\beta$.) If we note that $(\alpha\gamma+\beta^2)$ is simply the scalar product of the vectors representing the numbers $a_0$ and $a_{\pi/2}$, (27) may be written in the form

$$|a_0||a_{\pi/2}|\cos\tau \leqq -\frac{A}{2\pi}(\alpha-\gamma), \tag{28}$$

where $\tau$ is the angle between the two vectors.

It is of interest to note that, while (2) reduces to an equality only in the degenerate case when the domain $D$ coincides with $D_\theta$, (3) and its equivalents (27) and (28) becomes equalities for a nondegenerate class of domains. In particular, one can easily show, by actually computing $s_0$ and $s_{\pi/2}$, that the equality sign may be taken in (3), (27), and (28) if $D$ is taken to be the exterior of any ellipse.

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GRADUATE SCHOOL OF ENGINEERING, HARVARD UNIVERSITY