ON THE SUM OF CUBES

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Large capital letters A, B, $\cdots$ (without or with subscripts) will represent integers of the quadratic number field $Ra(\rho)$ where $\rho=(-1+(-3)^{1/2})/2$. Small latin letters a, b, $\cdots$ represent rational integers, and the conjugate of a number X is denoted by $\overline{X}$.

The object of this paper is to give a method for obtaining the complete rational integer solution for the diophantine equations of the form

$$\sum_{i=1}^{m} z_i^3 = 0, \qquad m > 3. \tag{1}$$

This equation with m even, $m=2n$, can be written as $\sum_{i=1}^{n}(X_i+\overline{X}_i)X_i\overline{X}_i=0$ where

$$X_i = z_{2i-1} + \rho(z_{2i-1} - z_{2i}) \tag{2}$$

and thus the problem of solving (1) in this case is reduced to that of finding all the integers x_i, X_i satisfying the equations

$$\sum_{i=1}^{n} x_i X_i \overline{X}_i = 0, \tag{3}$$

$$x_i = X_i + \overline{X}_i \qquad (i = 1, 2, \cdots, n) \tag{4}$$

and (2). When m is odd, $m=2n-1$, we solve the system (α) consisting of (3), $x_n=X_n=z_{2n-1}$ and (2), (4) for $i=1, 2, \cdots, n-1$.

The resolution of these two systems hinges on techniques developed by E. T. Bell [2],[1] being equivalent to the resolution of a system of multiplicative equations and a system of linear homogeneous equations in Ra in which the number of unknowns always exceeds the number of equations.

In solving (1) the following equations appear:

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0 \tag{5}$$

in which the x_i, y_i $(i=1, 2, \cdots, n)$ are $2n$ independent variables;

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0 \qquad (i = 1, 2, \cdots, m \leqq n - 1) \tag{6}$$

in which the n independent variables x_i are to be solved in terms of the coefficients a_{ij};

Presented to the Society, September 17, 1945; received by the editors June 23, 1947.

[1] Numbers in brackets refer to references cited at the end of the paper.

(7) $$A_1B_1C_1 = \cdots = A_{n-1}B_{n-1}C_{n-1}$$

in $3(n-1)$ independent variables A_i, B_i, C_i;

(8) $$p_1A_1\overline{A}_1 = p_2A_2\overline{A}_2 = \cdots = p_nA_n\overline{A}_n$$

in which the p_i, A_i are independent variables.

The solution [5, p. 20 (13)] of (5) is

$$x_i = aa_i, \qquad y_i = -\sum_{j=1}^{i-1} a_jb_{j,i} + \sum_{j=1}^{n-i} a_{i+j}b_{i,i+j}$$

for $i=1, 2, \cdots, n$ with the convention that a sum in which the lower limit exceeds the upper is vacuous. Note that there are $(n^2-n)/2$ free parameters $b_{j,k}$ and $n+1$ free parameters a, a_i.

If the system (6) is of rank m then its complete solution [3] in determinantal form is written down as follows. Let e_j be the determinant obtained by deleting the jth column from the matrix of the coefficients of the system consisting of (6) and the equations

$$c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n = 0 \qquad (i = 1, 2, \cdots, n-m-1)$$

in which the c_{ij} are arbitrary rational integers. Then

(9) $$x_j = (-)^j te_j/e \qquad (j = 1, 2, \cdots, n)$$

where t is an arbitrary integer and $e=(e_1, e_2, \cdots, e_n)$.

The system (7) is recursive and can be solved completely by the algorithm of reciprocal arrays [1] since the integers of $Ra(\rho)$ form a principal ideal ring. System (7) is equivalent to the equations

$$A_iB_iC_i = A_{n-1}B_{n-1}C_{n-1} \qquad (i = 1, 2, \cdots, n-2).$$

The solution of the typical equation is

$$\begin{aligned} A_i &= A_{i1}B_{i1}C_{i1}, & A_{n-1} &= A_{i1}H_{i1}F_{i1}, \\ B_i &= D_{i1}E_{i1}F_{i1}, & B_{n-1} &= D_{i1}B_{i1}J_{i1}, \\ C_i &= G_{i1}H_{i1}J_{i1}, & C_{n-1} &= G_{i1}E_{i1}C_{i1}. \end{aligned}$$

Then the values of A_{n-1} are equated, also those of B_{n-1}, and those of C_{n-1}. The resulting three systems are each of the type (7) with $n-2$ in place of $n-1$. By repetitions of the process the solution of (7) involving $(n-1)\equiv 3^{n-1}$ free parameters $K_1, K_2, \cdots, K_{(n-1)}$ is obtained in the form

$$A_i = K_1\Phi_i(K_4, \cdots, K_{(n-1)}), \qquad B_i = K_2\Psi_i(K_4, \cdots, K_{(n-1)}),$$
$$C_i = K_3\Theta_i(K_4, \cdots, K_{(n-1)})$$

where each of Φ_i, Ψ_i, Θ_i is a monomial in $3^{n-2}-1$ of the parameters $K_3, K_4, \cdots, K_{(n-1)}$ each occurring only once in a particular Φ_i, Ψ_i, Θ_i and

$$K_1K_2K_3\Phi_i(K_4, \cdots, K_{(n-1)})\Psi_i(K_4, \cdots, K_{(n-1)})\Theta_i(K_4, \cdots, K_{(n-1)}) = K_1K_2 \cdots K_{(n-1)}$$

for $i=1, 2, \cdots, n-1$.

The resolution of (8) is also recursive. This system is equivalent to the $n-1$ equations $p_nA_n\overline{A}_n = p_iA_i\overline{A}_i$ $(i = 1, 2, \cdots, n-1)$. The solution [4, Theorem 1] of the typical equation is

$$p_i = t_{i1}V_{i1}\overline{V}_{i1}, \qquad p_n = t_{i1}L_{i1}\overline{L}_{i1},$$
$$A_i = S_{i1}\overline{U}_{i1}L_{i1}, \qquad A_n = S_{i1}U_{i1}V_{i1}.$$

Then the values of p_n are equated and also those of A_n which yield the two independent systems

$$t_{i1}L_{i1}\overline{L}_{i1} = t_{n-1,1}L_{n-1,1}\overline{L}_{n-1,1} \tag{10}$$

and

$$S_{i1}U_{i1}V_{i1} = S_{n-1,1}U_{n-1,1}V_{n-1,1} \tag{11}$$

for $i=1, 2, \cdots, n-2$.

System (10) is of the type (8) with $n-1$ in place of n; system (11) is of the type (7) and its solution is therefore

$$S_{i1} = K_1\Phi_i(K_4, \cdots, K_{(n-1)}), \qquad U_{i1} = K_2\Psi_i(K_4, \cdots, K_{(n-1)}),$$
$$V_{i1} = K_3\Theta_i(K_1, \cdots, K_{(n-1)})$$

for $i=1, 2, \cdots, n-1$.

Hence all integral solutions of (8) are given by

$$A_i = K_1\Phi_i\overline{K}_2\overline{\Psi}_iL_{i1}, \qquad A_n = K_1\Phi_iK_2\Psi_iK_3\Theta_i,$$
$$p_i = t_{i1}K_3\overline{K}_3\Theta_i\overline{\Theta}_i, \qquad p_n = t_{i1}L_{i1}\overline{L}_{i1},$$

with the condition (10).

The process just applied to (8) is now repeated on (10) which will yield parametric expressions for t_{i1}, L_{i1} similar to those for p_i, A_i respectively subject to systems of the type (10) and (11) with $n-1$ replaced by $n-2$. We note that this process must finally yield the solutions in the form

$$A_i = KE_i, \qquad p_i = dM_i\overline{M}_i \tag{12}$$

where the E_i and also the M_i are products of integers of $Ra(\rho)$; all the E_i, and also all the M_i are of course not independent but they are each independent of K.

The resolution of (3) now follows. Applying (5) to (3) yields

$$X_i\overline{X}_i = aa_i, \qquad x_i = -\sum_{j=1}^{i-1} a_jb_{j,i} + \sum_{j=1}^{n-i} a_{i+j}b_{i,i+j}$$

for $i=1, 2, \cdots, n$.

Hence we must now solve the system of equations

$$X_i\overline{X}_i = aa_i \qquad (i = 1, 2, \cdots, n).$$

The solution of the typical equation is

$$X_i = p_iA_iB_i, \qquad a = p_iA_i\overline{A}_i, \qquad a_i = p_iB_i\overline{B}_i$$

in free parameters p_i, A_i, B_i.

Equating the value of a yields the system (8) and hence by (12) $A_i=KE_i$, $p_i=dM_i\overline{M}_i$ and therefore the complete solution of (3) is given by

$$(13) \qquad X_i = KR_i, \qquad x_i = -\sum_{j=1}^{i-1} a_jb_{j,i} + \sum_{j=1}^{n-i} a_{i+j}b_{i,i+j}$$

for $i=1, 2, \cdots, n$ where

$$(14) \qquad R_i = dM_i\overline{M}_iB_iE_i, \qquad a_i = dM_i\overline{M}_iB_i\overline{B}_i.$$

The resolution of (1) now follows. Put $K=k_1+\rho k_2$, $R_i=r_i+\rho s_i$; then $KR_i+\overline{K}\overline{R}_i=k_1(2r_i-s_i)-k_2(r_i+s_i)$. Then all the X_i, x_i satisfying (3) and (4) simultaneously are given by (13) where values are assigned to the parameters which determine R_i, a_i in (14) and then the $(n^2-n+4)/2$ unknowns k_1, k_2, b_{ij} are determined from the n linear homogeneous equations

$$(15) \quad k_1(2r_i - s_i) - k_2(r_i + s_i) + \sum_{j=1}^{i-1} a_jb_{j,i} - \sum_{j=1}^{n-i} a_{i+j}b_{i,i+j} = 0$$

for $i=1, 2, \cdots, n$, a system of the type (6).

Substitute this value of X_i in (2). Equate real and imaginary parts and all the rational integer solutions of (1) with $m=2n$ will be obtained.

To solve (1) when m is odd, $m=2n-1$, we proceed much as above, replacing system (15) by system (α) which is equivalent to (13) and the equations

$$k_1(2r_i - s_i) - k_2(r_i + s_i) + \sum_{j=1}^{i-1} a_jb_{j,i} + \sum_{j=1}^{m-i} a_{i+j}b_{i,i+j} = 0$$

for $i=1, 2, \cdots, n-1$,

$$k_1 r_n - k_2 s_n + \sum_{j=1}^{n-1} a_j b_{j,n} = 0, \qquad k_1 s_n + k_2(r_n - s_n) = 0,$$

a linear homogeneous system of $n+1$ equations in $(n^2-n+4)/2$ unknowns k_1, k_2, b_{ij}. The corresponding X_i given by (13) are substituted in (2) for $i=1, 2, \cdots, n-1$ and we put $X_n = z_{2n-1}$.

We conclude by exhibiting the complete solution of $\sum_{i=1}^{6} z_i^3 = 0$ in terms of integers of $Ra(\rho)$.

The complete solution of $p_1 A_1 \overline{A}_1 = p_2 A_2 \overline{A}_2 = p_3 A_3 \overline{A}_3$ is given by (12) where

$$M_1 = GHJT, \qquad M_2 = GFLN, \qquad M_3 = LPST,$$
$$E_1 = C\overline{DF}LNP\overline{QS}, \qquad E_2 = \overline{CDH}JPQST, \qquad E_3 = CDFGHJNQ,$$

and all the parameters are arbitrary. Hence from (14) we get the corresponding values of a_i, $R_i = r_i + \rho s_i$, where d, B_i are arbitrary. In this case (15) is a linear homogeneous system of 3 equations in 5 unknowns. To complete this linear system for resolution we adjoin the single equation

$$m_1 b_{23} + m_2 b_{13} + m_3 b_{12} + m_4 k_1 + m_5 k_2 = 0$$

with arbitrary coefficients m_i.

Hence with a_i, r_i, s_i as found above, (9) gives

$$ek_1 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(r_1 + s_1) + a_2(r_2 + s_2) + a_3(r_3 + s_3)),$$
$$ek_2 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(2r_1 - s_1) + a_2(2r_2 - s_2) + a_3(2r_3 - s_3)).$$

Then from (2) and (13) for $i=1, 2, 3$

$$z_{2i-1} = k_1 r_i - s_i k_2, \qquad z_{2i} = k_1(r_i - s_i) - k_2 r_i.$$

References

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