PARACOMPACTNESS AND PRODUCT SPACES

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A topological space is called *paracompact* (see [2])[1] if (i) it is a Hausdorff space (satisfying the T_2 axiom of [1]), and (ii) every open covering of it can be refined by one which is "locally finite" (=neighbourhood-finite; that is, every point of the space has a neighbourhood meeting only a finite number of sets of the refining covering). J. Dieudonné has proved [2, Theorem 4] that every *separable* metric (=metrisable) space is paracompact, and has conjectured that this remains true without separability. We shall show that this is indeed the case. In fact, more is true: paracompactness is identical with the property of *"full normality"* introduced by J. W. Tukey [5, p. 53]. After proving this (Theorems 1 and 2 below) we apply Theorem 1 to obtain a necessary and sufficient condition for the topological product of uncountably many metric spaces to be normal (Theorem 4).

For any open covering $\mathcal{W} = \{W_\alpha\}$ of a topological space, the *star* $(x, \mathcal{W})$ of a point x is defined to be the union of all the sets W_α which contain x. The space is *fully normal* if every open covering $\mathcal{V}$ of it has a "*Δ-refinement*" $\mathcal{W}$—that is, an open covering for which the stars $(x, \mathcal{W})$ form a covering which refines $\mathcal{V}$.

THEOREM 1. *Every fully normal T_1 space is paracompact.*

Let S be such a space, and let $\mathcal{V} = \{U_\alpha\}$ be a given open covering of S. (We must construct a locally finite refinement of $\mathcal{V}$. Note that S is normal [5, p. 49] and thus satisfies the T_2 axiom.)

There exists an open covering $\mathcal{V}^1 = \{U^1\}$ which Δ-refines $\mathcal{V}$, and by induction we obtain open coverings $\mathcal{V}^n = \{U^n\}$ of S such that $\mathcal{V}^{n+1}$ Δ-refines $\mathcal{V}^n$ ($n = 1, 2, \cdots$, to ∞). For brevity we write, for any $X \subset S$,

$$(1)\qquad \begin{aligned}(X, n) &= \text{star of } X \text{ in } \mathcal{V}^n\\ &= \text{union of all sets } U^n \text{ meeting } X\end{aligned}$$

(roughly corresponding to the "$1/2^n$-neighbourhood of X" in a metric space), and

$$(2)\qquad (X, -n) = S - (S - X, n).$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Thus, since the set (X, n) is evidently open, $(X, -n)$ is closed. Further, it is easily seen that

$$(3) \qquad (X, -n) = \{x \mid (x, n) \subset X\},$$

where (x, n), in conformity with (1), denotes the star of x in $\mathcal{U}^n$; and it readily follows that

$$(4) \qquad ((X, -n), n) \subset X.$$

From the definition of Δ-refinement we have

$$(5) \qquad ((X, n+1), n+1) \subset (X, n).$$

The trivial relations $X \subset Y \rightarrow (X, n) \subset (Y, n)$, $m \geqq n \rightarrow (X, m) \subset (X, n)$, $\overline{X} \subset (X, n)$, and $y \in (x, n) \leftrightarrow x \in (y, n)$ will also be useful.

For convenience, suppose the sets U_α of $\mathcal{U}$ are well-ordered.

Now define, for each α,

$$(6) \qquad V_\alpha^1 = (U_\alpha, -1), \; V_\alpha^2 = (V_\alpha^1, 2), \text{ and } V_\alpha^n = (V_\alpha^{n-1}, n) \qquad (n \geqq 2).$$

Thus $V_\alpha^1 \subset V_\alpha^2 \subset \cdots$, and V_α^n is open if $n \geqq 2$; hence, writing $\bigcup_n V_\alpha^n = V_\alpha$, we have that V_α is open. An easy induction (using (4) and (5)) shows that $(V_\alpha^n, n) \subset U_\alpha$; hence

$$(7) \qquad V_\alpha \subset U_\alpha.$$

Further,

$$(8) \qquad \bigcup V_\alpha = S,$$

since, given $x \in S$, we have $(x, 1) \in$ some U_α (for $\mathcal{U}^1$ Δ-refines $\mathcal{U}$), so that, by (3), $x \in V_\alpha^1 \subset V_\alpha$.

We also have

(9) Given $x \in V_\alpha$, there exists $n > 0$ such that $(x, n) \subset V_\alpha$.

For there exists $n \geqq 2$ such that $x \in V_\alpha^{n-1}$, and then $(x, n) \subset V_\alpha^n \subset V_\alpha$.

Next we define, for each $n > 0$, a transfinite sequence of closed sets $H_{n\alpha}$, by setting

$$(10) \qquad H_{n1} = (V_1, -n), \qquad H_{n\alpha} = \left(V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}, -n\right).$$

Then we have:

(11) If $\alpha \neq \gamma$, no $U^n \in \mathcal{U}^n$ can meet both $H_{n\alpha}$ and $H_{n\gamma}$.

For we can suppose $\gamma < \alpha$. Then if U^n meets $H_{n\alpha}$, let $x \in U^n \cap H_{n\alpha}$; from (3) and (10), $U^n \subset V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}$, and so is disjoint from $H_{n\gamma}$.

(12) $$\bigcup_{n,\alpha} H_{n\alpha} = S.$$

For, given $x \in S$, (8) shows that there will be a *first* ordinal α for which $x \in V_\alpha$; and from (9) there exists $n > 0$ such that $(x, n) \subset V_\alpha$. We assert $x \in H_{n\alpha}$. For suppose not. Then, from (10) and (3), (x, n) contains a point y not in $V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}$; and it follows that $y \in H_{n\beta}$ for some $\beta < \alpha$. But then $x \in (H_{n\beta}, n) \subset ((V_\beta, -n), n) \subset V_\beta$ (from (4)); and this contradicts the definition of α.

Write

(13) $$E_{n\alpha} = (H_{n\alpha}, n + 3), \qquad G_{n\alpha} = (H_{n\alpha}, n + 2).$$

Thus $H_{n\alpha} \subset E_{n\alpha} \subset \overline{E_{n\alpha}} \subset G_{n\alpha}$, and, as is easily seen from (11),

(14) If $\gamma \neq \alpha$, no $U^{n+2} \in \mathcal{U}^{n+2}$ can meet both $G_{n\alpha}$ and $G_{n\gamma}$.

Write $F_n = \bigcup_\alpha \overline{E_{n\alpha}}$. Then F_n *is closed.* For suppose $x \in \overline{F}_n$. Then every open neighbourhood $N(x)$ of x meets some $\overline{E_{n\alpha}}$ and so meets some $E_{n\alpha}$; but if $N(x)$ is contained in the neighbourhood $(x, n+2)$ of x, $N(x)$ can meet at most *one* set $E_{n\alpha}$ (from (14)), so that $x \in \overline{E_{n\alpha}} \subset F_n$.

Finally we define

$$W_{1\alpha} = G_{1\alpha}, \qquad W_{n\alpha} = G_{n\alpha} - (F_1 \cup F_2 \cup \cdots \cup F_{n-1}) \qquad (n > 1);$$

thus the sets $W_{n\alpha}$ are open. We shall show that they form the desired refinement.

In the first place, $\bigcup_{n,\alpha} W_{n\alpha} = S$. For, given $x \in S$, we have $x \in$ some $H_{n\alpha}$ (from (12)) $\subset \overline{E_{n\alpha}}$; let m be the smallest integer for which there exists $\overline{E_{m\beta}} \ni x$. Then $x \in G_{m\beta}$, and $x \notin F_1 \cup \cdots \cup F_{m-1}$, so that $x \in W_{m\beta}$.

Next, $W_{n\alpha} \subset G_{n\alpha} \subset (H_{n\alpha}, n) \subset ((V_\alpha, -n), n) \subset V_\alpha \subset U_\alpha$ (using (4) and (7)). Thus the sets $W_{n\alpha}$ form an open covering $\mathcal{W}$ of S which refines $\mathcal{U}$. All that remains to be proved is that $\mathcal{W}$ is "locally finite." Given $x \in S$, we have as before that $x \in$ some $H_{n\alpha}$, so $(x, n+3) \subset E_{n\alpha} \subset F_n$, and so is certainly disjoint from $W_{k\beta}$ if $k > n$. Further, for a given $k \leqq n$, we have $(x, n+3) \subset U^{n+2} \subset U^{k+2}$, so (13) shows that $(x, n+3)$ can meet $W_{k\beta}$ for at most *one* value of β. Thus the neighbourhood $(x, n+3)$ of x meets at most n of the sets $W_{k\beta}$; and the proof is complete.

REMARK. The locally finite refinement $\mathcal{W}$ thus constructed has the additional property that it consists of a countable number of families of sets (formed by the sets $W_{n\alpha}$, n fixed), the sets of each family having pairwise disjoint closures.

COROLLARY 1. *Every metric space is paracompact.*

For a metric space is fully normal [5, p. 53].

COROLLARY 2. *The topological product of a metric space and a com-*

pact (=bicompact) Hausdorff space is paracompact, and therefore normal.[2]

This follows from Theorems 5 and 1 of [2].

THEOREM 2. *Every paracompact space is fully normal (and T_1).*

Let S be a paracompact space, and let $\mathcal{U}=\{U_\alpha\}$ be a given *locally finite* open covering of S. It will evidently suffice to prove that $\mathcal{U}$ has a Δ-refinement.

Open sets X_α exist, for each α, such that $\overline{X}_\alpha \subset U_\alpha$ and $\bigcup X_\alpha = S$. (This follows by an easy transfinite induction argument from the fact that S is normal; cf. [2, Theorems 1 and 6].) By hypothesis, each $x\in S$ has an open neighbourhood $V(x)$ meeting U_α only for a finite set of α's, say for $\alpha\in A(x)$. Let $B(x)$ be the set of those α's $\in A(x)$ for which $x\in U_\alpha$, and let $C(x)$ be the set of α's $\in A(x)$ for which $x\notin \overline{X_\alpha}$; clearly $B(x)\cup C(x)=A(x)$. Define

$$W(x) = V(x) \cap \bigcap\{U_\alpha \mid \alpha \in B(x)\} \cap \bigcap\{(S - \overline{X}_\alpha) \mid \alpha \in C(x)\}.$$

Evidently $W(x)$ is an open set containg x; hence the sets $\{W(x) \mid x\in S\}$ form an open covering $\mathcal{W}$ of S. To verify that $\mathcal{W}$ is a Δ-refinement of $\mathcal{U}$, let $y\in S$ be given. There exists a set $X_\beta \ni y$; we shall show that the star $(y, \mathcal{W})\subset U_\beta$—that is, that if $y\in W(x)$ then $W(x)\subset U_\beta$. For if $y\in W(x)$ then $W(x)$ meets $\overline{X}_\beta$ and so clearly $\beta\in A(x)$ and $\beta\notin C(x)$. Thus $\beta\in B(x)$, which implies $W(x)\subset U_\beta$, by construction.

Now let N denote the space of positive integers—a countable discrete set—and consider the *product* $T=\prod N_\lambda$ $(\lambda\in\Lambda)$ of uncountably many copies of N. More precisely, the points of T are the mappings $x=\{\xi_\lambda\}$ of the uncountable set Λ in N (each $\lambda\in\Lambda$ being mapped on the integer $\xi_\lambda\in N$), and a typical basic neighbourhood U of x in T is obtained by choosing a *finite* set $\mathcal{R}(U)\subset\Lambda$ and defining U to consist of all points $y=\{\eta_\lambda\}$ such that $\eta_\lambda=\xi_\lambda$ for all $\lambda\in\mathcal{R}(U)$. $\mathcal{R}(U)$ will be called the "set of coordinates *restricted* in U."

THEOREM 3. *The space T is not normal.*

For each positive integer k, let A^k be the set of all points $x=\{\xi_\lambda\}$ $\in T$ satisfying: for each positive integer n other than k, there is at most one λ for which $\xi_\lambda = n$.

[2] It can be shown that the topological product of a metric space and a normal countably compact space is normal, though not necessarily paracompact. (A space is "countably compact" if every infinite subset has a limit point in the space; cf. [5, p. 42]. For metric spaces this is equivalent to compactness.)

It is easy to see that the sets A^k are closed and pairwise disjoint. Hence, if T were normal, there would exist disjoint open sets U, V such that $U\supset A^1$, $V\supset A^2$. We shall show that this leads to a contradiction.

We shall define inductively sequences of points $x_n\in A^1$, of integers $0<m(1)<m(2)<\cdots$, and of elements $\lambda_n\in\Lambda$, as follows. Define x_1 to be the point $\{\xi_\lambda\}$ for which $\xi_\lambda=1$ (all $\lambda\in\Lambda$). Evidently $x_1\in A^1\subset U$, so x has a basic neighbourhood $U_1\subset U$. Let $\mathfrak{R}(U_1)$ consist of the $m(1)$ elements λ_k $(1\leqq k\leqq m(1))$. When x_α and $\lambda_1, \lambda_2 \cdots, \lambda_{m(n)}$ have been defined, in such a way that $x_n\in A^1$ and $\lambda_1, \cdots, \lambda_{m(n)}$ are the coordinates restricted in a basic neighbourhood $U_n\subset U$ of x_n, we define x_{n+1} by: $\xi_\lambda=k$ if $\lambda=\lambda_k$ $(1\leqq k\leqq m(n))$, and $\xi_\lambda=1$ otherwise. Clearly $x_{n+1}\in A^1$, so that x_{n+1} has a basic neighbourhood $U_{n+1}\subset U$; and we can always suppose that $\mathfrak{R}(U_{n+1})$ contains $\mathfrak{R}(U_n)$ as a proper subset. Let $\mathfrak{R}(U_{n+1})$ have $m(n+1)$ elements, and enumerate the elements of $\mathfrak{R}(U_{n+1})-\mathfrak{R}(U_n)$ as $\lambda_{m(n)+1}, \cdots, \lambda_{m(n+1)}$. The induction is now complete.

Now define a point $y=\{\eta_\lambda\}$ by: $\eta_\lambda=k$ if $\lambda=\lambda_k$ $(k=1, 2, \cdots, \text{to } \infty)$ and $\eta_\lambda=2$ otherwise. Clearly $y\in A^2\subset V$, so y has a basic neighbourhood $V_0\subset V$. Since $\mathfrak{R}(V_0)$ is finite, there exists an n such that $\lambda_k\in\Lambda-\mathfrak{R}(V_0)$ whenever $k>m(n)$. Finally, define $z=\{\zeta_\lambda\}$ by:

$$\begin{aligned}
\zeta_\lambda &= k \quad \text{if} \quad \lambda = \lambda_k \text{ with } k \leqq m(n),\\
\zeta_\lambda &= 1 \quad \text{if} \quad \lambda = \lambda_k \text{ with } m(n) < k \leqq m(n+1), \quad \text{and}\\
\zeta_\lambda &= 2 \quad \text{otherwise.}
\end{aligned}$$

We evidently have $z\in U_{n+1}\cap V_0\subset U\cap V$, giving the desired contradiction.

COROLLARY. *If a product of nonempty T_1 spaces is normal, all but at most a countable number of the factor spaces must be countably compact.*[2]

For otherwise their product would contain a closed subset homeomorphic with T; and a closed subset of a normal space is normal.

THEOREM 4. *The following statements about a product of nonempty metric spaces are equivalent.*

(i) *The product is normal.*

(ii) *The product is fully normal* (*or paracompact*).

(iii) *At most* $\aleph_0$ *of the factor spaces are noncompact.*

In fact, (ii)→(i) [2, Theorem 1], (i)→(iii) (Theorem 3, Corollary), and (iii)→(ii) from Theorem 1, Corollary 2, since the compact