THE FEBRUARY MEETING IN NEW YORK

The 466th meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, February 24, 1951. The meeting was attended by over 200 persons, including the following 189 members of the Society:

Milton Abramowitz, M. I. Aissen, E. J. Akutowicz, C. B. Allendoerfer, R. D. Anderson, R. L. Anderson, T. W. Anderson, R. G. Archibald, M. C. Ayer, K. Y. Bal, Iacopo Barsotti, M. F. Becker, E. G. Begle, H. B. Belck, A. H. Berger, Stefan Bergman, Lipman Bers, Nicholas Bilotta, D. W. Blackett, W. E. Bleick, Raoul Bott, Samuel Bourne, A. D. Bradley, F. E. Browder, C. T. Bumer, J. H. Bushey, Jewell H. Bushey, F. P. Callahan, W. R. Callahan, H. E. Campbell, P. G. Carlson, Paul Chessin, Joshua Chover, W. L. Chow, K. L. Chung, F. E. Clark, E. A. Coddington, L. W. Cohen, R. M. Cohn, Philip Cooperman, Natalie Coplan, L. M. Court, W. H. H. Cowles, H. F. Cullen, P. M. Curran, J. H. Curtiss, Vittorio Dalla Volta, G. B. Dantzig, M. D. Darkow, R. J. De Vogelaere, J. A. Dieudonné, Avron Douglis, Nelson Dunford, Aryeh Dvoretzky, Joanne Elliott, J. M. Feld, W. E. Ferguson, F. A. Ficken, D. T. Finkbeiner, R. S. Finn, J. B. Freier, Gerald Freilich, Bernard Friedman, T. C. Fry, Bent Fuglede, David Gale, Abe Gelbart, Irving Gerst, B. P. Gill, G. H. Gleissner, Sidney Glusman, H. E. Goheen, Oscar Goldman, Laura Guggenbuhl, Harish-Chandra, G. A. Hedlund, Alex Heller, Erik Hemmingsen, A. D. Hestenes, T. W. Hildebrandt, W. M. Hirsch, A. J. Hoffman, S. P. Hoffman, Banesh Hoffmann, Heinz Hopf, S. T. Hu, E. M. Hull, T. R. Humphreys, L. C. Hutchinson, W. S. Jardetzky, M. L. Juncosa, R. V. Kadison, Shizuo Kakutani, S. N. Karp, M. E. Kellar, D. E. Kibbey, J. F. Kiefer, H. S. Kieval, Morris Kline, E. G. Kogbetliantz, E. R. Kolchin, Saul Kravetz, Wouter van der Kulk, Benjamin Lepson, Howard Levi, Norman Levinson, Charles Loewner, E. R. Lorch, S. S. McNeary, L. A. MacColl, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, Murray Mannos, A. J. Maria, M. H. Maria, H. G. Mazurkiewicz, A. N. Milgram, Joseph Milkman, J. M. Miller, K. S. Miller, W. H. Mills, C. R. Morris, T. S. Motzkin, W. R. Murray, D. S. Nathan, J. D. Newburgh, Katsumi Nomizu, P. B. Norman, A. M. Peiser, R. S. Phillips, E. L. Post, Walter Prenowitz, F. M. Pulliam, J. F. Randolph, H. E. Rauch, Moses Richardson, R. E. Roberson, J. H. Roberts, Robin Robinson, S. L. Robinson, I. H. Rose, Saul Rosen, R. A. Rosenbaum, Maxwell Rosenlicht, J. E. Rosenthal, H. D. Ruderman, J. P. Russell, Bernard Sachs, James Sanders, Arthur Sard, L. R. Sario, A. T. Schafer, Henry Scheffé, Leonard Schieber, Pincus Schub, Abraham Schwartz, B. L. Schwartz, Seymour Sherman, James Singer, P. A. Smith, J. J. Sopka, G. L. Spencer, F. M. Stewart, Walter Strodt, R. L. Swain, P. M. Treuenfels, C. A. Truesdell, J. W. Tukey, Annita Tuller, A. H. Van Tuyl, D. H. Wagner, R. M. Walter, W. R. Wasow, M. A. Weber, J. V. Wehausen, H. F. Weinberger, B. A. Welch, David Wellinger, M. E. Wells, J. G. Wendel, M. E. White, Albert Wilansky, Jacob Wolfowitz, Albert Wolinsky, E. S. Wolk, M. A. Woodbury, N. J. Zabb, H. J. Zimmerberg.

Professor Heinz Hopf of the Swiss Federal School of Technology and Princeton University delivered an invited address *On complex and almost complex manifolds* at the general session presided over by Professor G. A. Hedlund. It was announced that the Duke Mathematical Journal had lifted its "moratorium" on the acceptance of manuscripts.

The sessions for contributed papers in the morning were presided over by Dr. A. D. Hestenes and Professor Bernard Friedman. Dr. T. S. Motzkin presided at the afternoon session.

Abstracts of the papers presented are listed below, those with a "*t*" after their numbers having been read by title. Papers numbered 188, 213, 215 were presented by Professor Gale, Dr. Gerst, and Dr. Motzkin, respectively. Mr. Gaier was introduced by Professor J. F. Randolph.

ALGEBRA AND THEORY OF NUMBERS

202*t*. Eckford Cohen: *A theorem on polynomial sums.*

Let r be a positive integer and let A, B, F be polynomials over $GF(p^n)$ of degree less than r with A, B primary, $(A, B)=1$. The author considers the problem of determining the number of solutions N of $F=AU+BV$ in polynomials U, V with deg $AU<r$, deg $BV<r$. Placing $a=\deg A$, $b=\deg B$, $r=a+b+s$, the following result is proved: Let X denote the unique polynomial of degree less than a satisfying $BX\equiv 1 \pmod{A}$. Then if $s\geqq 0$ it follows that $N>0$ if and only if deg $(FX \pmod{A})<a-s$, in which case $N=1$. If $s<0$, then $N=p^{-ns}$. The result for $s<0$ was proved earlier (Duke Math. J vol. 14 (1947) pp. 253–254); the present proof is based on the fundamental theorem of arithmetic functions (L. Carlitz, Duke Math. J. vol. 14 (1949) p. 1124). (Received January 10, 1951.)

203*t*. Harvey Cohn: *Stable lattices.*

The author unifies some of his earlier work (Bull. Amer. Math. Soc. Abstracts 57-1-15, 57-1-16, and 57-2-82), with recent researches of Mahler on star bodies and earlier researches of Korkine and Zolatareff. Let $x_i=\sum a_{ij}m_j$ be a lattice, m_j integral, with $\det=|a_{ij}|>0$, $1\leqq i, j\leqq n$. Let $\phi(x_i)$ be a homogeneous function of degree h. Define $F^{(k)}=F(m_j^{(k)}; a_{ij})=\phi(x_i^{(k)})/|a_{ij}|^{h/n}$ for a set of n-tuples $(m_j^{(k)})$. Consider the differentials $d|F^{(k)}|$ in the variables a_{ij} about a fixed value of a_{ij}. They form a set whose dimension is determined by the set of $(A^{(k)})$ for which $\sum A^{(k)}d|F^{(k)}|=0$. If the variable k indexes *all* n-tuples, the dimension is independent of the lattice and is called Q, the *free dimension*. If a set of $Q+1$ n-tuples are such that $|F^{(k)}|$ equals its minimum $\mu(a_{ij})$ for the lattice and such that the (one-dimensional) family of $A^{(k)}$ are proportional to positive constants, then the lattice is called *stable*. The $A^{(k)}$ are also determined by $\sum A^{(k)}R_{ij}^{(k)}=0$, $1\leqq i, j\leqq n$, where $R_{ij}=x_i\partial|\phi|/\partial x_j-(h/n)|\phi|\delta_{ij}$. A measure of the distance between two lattices can be defined so that for stable lattices (a_{ij}^*), for constants ϵ and ρ, $\mu(a_{ij}^*)\leqq\mu(a_{ij})-\rho\|a_{ij}-a_{ij}^*\|$ if $\|a_{ij}-a_{ij}^*\|<\epsilon$. The last result has a partial converse. Minkowski's extremal lattices are in general not stable. (Received January 8, 1951.)

204. David Gale and F. M. Stewart: *Infinite games with complete information.*

Von Neumann and Morgenstern have defined the notion of a finite, two-person, zero-sum game with complete information and have proved the fundamental theorem to the effect that all such games are strictly determined. It is shown here that the theorem no longer holds if the game is allowed to be infinite. The following simple example illustrates the situation: The unit interval $[0, 1]$ is partitioned into two sets S_{I} and S_{II}. Players I and II alternately pick a digit from 0 to 9 thus giving rise to an infinite sequence of digits which corresponds in an obvious way to an infinite decimal, hence a number on $[0, 1]$. The winner is player I or player II as this number lies in S_{I} or S_{II}. It is shown that S_{I} and S_{II} may be so chosen that this game is not strictly determined! In the other direction some general theorems on infinite games show that if S_{I} and S_{II} are sufficiently well-behaved (for example, open or closed sets) then the game is determined. (Received December 15, 1950.)

205*t*. Oscar Goldman: *Hilbert rings and the Hilbert Nullstellensatz.*

For commutative rings with unit element, two notions of radical have been defined: the ideal of nilpotent elements and the intersection of the maximal ideals. A ring R is called a *Hilbert ring* if for every ideal I in R, the two notions of radical coincide in R/I. A homomorphic image of a Hilbert ring is one also. Furthermore, R is a Hilbert ring simultaneously with $R[x]$, x transcendental over R. From this it follows that $k[x_1, \cdots, x_n]$ is a Hilbert ring, when k is a field. An important characterization of this class of rings is the following: R is a Hilbert ring if, and only if, every maximal ideal in $R[x]$ contracts to a maximal ideal in R. This implies immediately the Hilbert Nullstellensatz. (Received November 21, 1950.)

206*t*. Stanley Katz: *On the representation of powerfree integers by systems of polynomials.*

Take c integers $r_i \geqq 2$, and corresponding polynomials $f_i(x)$ with integral coefficients, $1 \leqq i \leqq c$. Let each $f_i(x)$ have as divisors exactly γ_i distinct monic rationally irreducible polynomials $g_{ij}(x)$ of positive degree with rational coefficients, where $g_{ij}(x)$ is of degree t_{ij} and appears in the factorization of $f_i(x)$ to the power d_{ij}, $1 \leqq j \leqq \gamma_i$. Set $\sigma = \max_{1 \leqq i \leqq c, 1 \leqq j \leqq \gamma_i} \{t_{ij}/-[-r_i/d_{ij}]\}$, where $-[-\theta]$ is the smallest integer greater than or equal to θ. Denote by $\mu_r(N)$ that function of the integer N which is 0 if N is divisible by the rth power of a prime and 1 otherwise. Then, if $\phi(n)$ is any complex-valued function of the integer n with positive integral period, it is shown that $\sum_{1 \leqq n \leqq z} \phi(n) \prod_{1 \leqq i \leqq c} \mu_{r_i}\{f_i(n)\} = \lambda z + o(z)$ as $z \to \infty$, provided $\sigma \leqq 1$. The constant λ is given explicitly by an absolutely convergent series, and estimates are given for the error term $o(z)$. Also, if the periodic function $\phi(n)$ is real and non-negative, it is shown that $\lambda > 0$ if and only if $\phi(\nu) \prod_{1 \leqq i \leqq c} \mu_{r_i}\{f_i(\nu)\} > 0$ for some integer ν, provided again that $\sigma \leqq 1$. (Received January 5, 1951.)

ANALYSIS

207. M. I. Aissen: *Cyclically-ordered sets.*

The classical separation theorems concerning the zeros of pairs of consecutive orthogonal polynomials are generalized. The methods involve a Sturm-type analysis of certain linear difference equations. The concept of cyclically-ordered sets is introduced to simplify the proofs. A family of k finite sets of real numbers is said to be *cyclically-ordered* if and only if they are disjoint in pairs, and if any k consecutive elements of their union contain one element from each of the sets. Some of the theorems proved are the following. In a set of symmetric orthogonal polynomials (each polynomial is either an odd or an even function) the sets consisting of the *positive* zeros of any *three* consecutive polynomials are cyclically-ordered. If we define the polynomials $p_n^{(k)}(t)$ by the generating function $\exp(x + tx^k) = \sum_{n=0}^{\infty} p_n^{(k)}(t) x^n$, then the sets consisting of the zeros of any $k+1$ consecutive polynomials (k is fixed) are cyclically-ordered. This is used to show that, for fixed k, if $p_n^{(k)}(t) \geqq 0$ for all n, then $t \geqq 0$. (Received January 15, 1951.)

208*t*. Stefan Bergman: *Representation of singular solutions of a linear differential equation.*

The author considers a solution of an equation of the form $L(\psi) \equiv \Delta\psi + 4N(\lambda)\psi = 0$, where N belongs to a certain class. Let $\lambda_0 < 0$, and $\lambda' = \lambda - \lambda_0$. If, for a solution $\psi(\lambda', \theta)$, $\psi(0, \theta) = \kappa_1(\theta) = \sum \beta_n \theta^n$ and $\psi_{\lambda'}(0, \theta) = \kappa_2(\theta) = \sum \gamma_n \theta^n$ are given, the author determines the associate $g = \sum (\mu_{n,1} + i\mu_{n,2}) Z'^n$, $Z' = \lambda' + i\theta$, of the second kind. One then obtains $\mu_{2n,1} = (-1)^n \beta_{2n}$, $\mu_{2n+1,2} = (-1)^{n+1} \beta_{2n+1}$, $\mu_{2n+2-k,k} = \sum_{\nu=0}^{n} D_\nu S_\nu$, $k = 1, 2$, $D_\nu = (-1)^n \gamma_{2n/(2n+1)} - \sum_{j=1}^{n} S_j^* \beta_{2n-2j}$, where S_n and S_ν^* are certain constants which depend upon N. ψ is regular in every domain $B \in [(\lambda - \lambda_0)^2 + \theta^2 < 4\lambda^2, \lambda < 0]$ in which g is regular. If g has poles or branchpoints, in B, ψ has pole-like singularities or branch points of the second kind. If the $\mu_{n,k}$, $k = 1, 2$, satisfy certain conditions of the Hadamard type, ψ can be represented in B in the form $\psi = \int_{t=-1}^{1} E(Z', \bar{Z}', t)[f_1(Z'(1-t^2)/2) \cdot (f_2(Z'(1-t^2)/2))^{-1}] dt$, where f_1 and f_2 are analytic functions which are regular in B, and E (the generating function of the operator) depends only on N. (Received February 23, 1951.)

209*t*. Stefan Bergman: *The initial value problem in the large for differential equations with singular coefficients.*

Using the integral operator of the second kind, the initial value problem in the large for the solutions of the equations $L(\psi) \equiv \Delta\psi + 4N(\lambda)\psi = 0$ is considered. Here, $N(\lambda) = (-\lambda)^{-1}[-(1/12) + \sum_{\nu=1}^{\infty} B_\nu (-\lambda)^{2/3\nu}]$, $B_1 \geqq 0$, is a (real) analytic function for $-\infty < \lambda < 0$, such that $[-\int_{-\infty}^{\lambda} N(t) dt]$ exists for all $\lambda < 0$, and satisfies some other additional conditions. The author determines from the given initial data $\psi(0, \theta) = \chi_1(\theta)$, $\lim_{\lambda \to 0}[(-\lambda)^{1/3} \partial(\lambda, \theta)/\partial\lambda] = \chi_2(\theta)$, the domain B situated in $[|\lambda| < 3^{1/2}|\theta|, \lambda < 0]$, where the solution $\psi(\lambda, \theta)$ is regular. This domain depends only on χ_1 and χ_2, but is independent of the B_ν's. The author also gives sufficient conditions that ψ has singularities on some points of the boundary of B. The proof consists in reducing the above problems to the question of determining the regularity domain of an analytic function $f(z) = \sum a_n z^n$ from the coefficients a_n. These results can also be interpreted as theorems for the behavior of the solution of the initial value problem in the large for certain equations of mixed type, with the initial data given on the transition line. (Received February 23, 1951.)

210*t*. S. D. Bernardi: *A new upper bound for the fourth coefficient of a schlicht function.*

Let (S) denote the class of functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$, which are regular and schlicht for $|z| < 1$. It is proven that if $f(z) \subset (S)$ then $|a_4| < 4.0891$. This is an improvement upon Friedman's estimate (see B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. vol. 13 (1946)) of $|a_4| < 4.16$. The method used is as follows: Let $f(z) \subset (S)$. Form $[f(z)/z]^{-\alpha/2} = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then by Prawitz (see H. Prawitz, *Über Mittlewerte analytischer Funktionen*, Arkiv for Matematik, Astronomi och Fysik vol. 20 (1927–1928)) it follows that $\sum_{n=1}^{\infty} (2n - \alpha)|b_n|^2/\alpha \leqq 1$, $\alpha > 0$. The author considers $\sum_{n=1}^{4} (2n-2)|b_n|^2/2 = H_1(a_2, a_3, \bar{a}_3, a_4, a_i) \leqq 1$. By a result by Marty (see F. Marty, *Sur le module des coefficients de MacLaurin d'une fonction univalente*, C. R. Acad. Sci. Paris vol. 198 (1934) pp. 1569–1571) if $f(z) \subset (S)$ and has maximum $|a_n|$, then $(n+1)a_{n+1}-2a_2a_n=(n-1)\bar{a}_{n-1}$; letting $n=4$ the author considers $H_1(a_2, a_3, \bar{a}_3, a_4, 2/5a_2a_4-3/5\bar{a}_3)\leqq 1$. Solving for $|a_4|$, the author proves that $|a_4|\leqq H_2(a_2, a_3, \bar{a}_3)$ takes its maximum value when both a_2 and a_3 are real, and hence proceeds to maximize $|a_4|=H_3(a_2, a_3)$ over the Peschl region (see E. Peschl, *Zur Theorie der schlichten Funktionen*, J. Reine Angew. Math. vol. 176 (1937) pp. 61–94) of variability of (a_2, a_3). (Received December 26, 1950.)

211*t*. S. D. Bernardi: *Schlicht functions with integral coefficients in a quadratic field.*

Let $K((-N)^{1/2})$ be the imaginary quadratic extension generated over the rational field by a root of $x^2+N=0$, where $N\geqq 1$ is a rational integer having no square factor. Let $K(1)$. denote the field of rationals. Let (S) denote the class of functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1=1$, which are regular and schlicht for $|z|<1$. Let $S[K((-N)^{1/2})]$ denote the class of functions $f(z)\subset(S)$ whose coefficients are algebraic integers in $K((-N)^{1/2})$. It is proven that the class $S[K((-N)^{1/2})]$ is finite for $N\geqq 1$ and consists of exactly forty-five functions of the form $f(z)=z/1-a_2z+(a_2^2-a_3)z^2$, where $|a_2^2-a_3| =0$ or 1, and the roots of $H(z)=1-a_2z+(a_2^2-a_3)z^2=0$ lie among the nonprimitive 24th roots of unity. These functions are explicitly given. The existence of the functions follows from the generalized "area principle" of Prawitz (see H. Prawitz, *Über Mittlewerte analytischer Funktionen*, Arkiv for Matemik, Astronomi och Fysik vol. 20 (1927–1928)), while the proofs for uniqueness follow the methods of Friedman (see B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. vol. 13 (1946)) who found the class $S[K(1)]$. All results apply equally well to the larger class of mean 1-valent functions. For real quadratic fields the class $S[K(-N^{1/2})]$ is infinite for each $N\geqq 2$. (Received December 18, 1950.)

212. R. S. Finn: *A new proof and generalization of a result of Bers.*

A simple criterion on the coefficients is used to characterize a class Ψ of quasilinear elliptic partial differential equations of the form $(\rho\phi_x)_x+(\rho\phi_y)_y=0$ for which a solution, single-valued in a domain D, admits only removable isolated singularities in D. The case $\rho=(1+\phi_x^2+\phi_y^2)^{-1/2}$ has been treated by Bers (Bull. Amer. Math. Soc. Abstract 54-7-239). It is shown that this criterion distinguishes a broad class Ω of nonlinear equations, for which the behavior of a single-valued solution at an isolated singular point is severely restricted. In particular, such a function is bounded at every isolated singular point. This is true also for solutions of equations of a class $\Omega', \Psi\subset\Omega' \subset\Omega$, which are finitely multivalued in a neighborhood of an isolated singular point. (Received November 13, 1950.)

213. Deiter Gaier: *Behavior of power series on the border of summability.*

If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is convergent for $z=1$ to the sum s, a well known Abelian theorem says that $f(z)\rightarrow s$ for $z\rightarrow 1-0$. The author proved some theorems, which include extensions of this Abelian theorem in three directions: (1) Instead of the circle of convergence he takes the border of summability for some transformation methods. (2) Some kinds of complex approach are admitted in addition to real-axis approach. (3) Instead of concluding a convergent behavior of $f(z)$ from convergent behavior of the series he assumes more generally a certain asymptotic behavior of the transformation of the series and infers an asymptotic behavior of $f(z)$. For the transformation of the series the methods of Cesàro, Euler, Borel, and Meyer-König are used.

Similar results are true for Dirichlet's series and Laplace integrals, for which the methods of Riesz and Cesàro are used, respectively. (Received December 27, 1950.)

214t. J. J. Gergen and F. G. Dressel: *Mapping by p-regular functions.*

Let Γ denote a circle with interior S. If $W(z)\in C'$ on S, $z=x+iy$, and if there exists a constant M such that $|W_x+iW_y|\leqq M|W|$ on S, then $W\in\mathfrak{M}$ on S. The following are examples of results obtained for functions of class $\mathfrak{M}$. Let $W\in\mathfrak{M}$ on S and C^0 on $\bar{S}=S+\Gamma$; if $W\not\equiv 0$ on S there corresponds to each zero z_0 of W in S a positive integer $k=k(z_0)$ and a number $L=L(z_0)\neq 0$ such that $W/(z-z_0)^k\to L$ $z\to z_0$; if $W\equiv 0$ on an arc of Γ, then $W\equiv 0$ on $\bar{S}$. By means of properties of functions of class $\mathfrak{M}$ the authors extend their previous mapping results for p-regular functions (see Bull. Amer. Math. Soc. Abstract 54-11-464) to the case $p\in C^0$ on $\bar{S}$, $p>0$ on $\bar{S}$, p_x, p_y exist and satisfy uniform Hölder conditions on S. (Received January 8, 1951.)

215. H. E. Goheen: *On the two-point boundary value problem for the equation $u''=f(u)$.*

The author discusses the two-point boundary value problem for the nonlinear ordinary differential equation $u''=f(u)$. Results include a bound for the magnitude of the solution in the interval and a rather remarkable inequality on the gamma function. (Received January 3, 1951.)

216. R. V. Kadison: *Order properties of bounded self-adjoint operators.*

It is well known that the self-adjoint operators in a commutative C^*-algebra (that is, a uniformly closed self-adjoint algebra of operators on a Hilbert space) form a lattice in the usual operator order. S. Sherman [to appear in the Amer. J. Math.] has shown the converse to be true. Some examples were known of pairs of self-adjoint operators which did not have a greatest lower bound in all bounded operators. It is shown that the only time two self-adjoint operators have a greatest lower bound is when they are comparable (that is, one is greater than or equal to the other). A partially ordered system with the property that two elements have a greatest lower bound in the system only if they are comparable is called "an anti-lattice." It is proved more generally that the class of factors (weakly closed C^*-algebras whose centers consist of multiples of the identity operator) is identical with the class of weakly closed C^*-algebras whose self-adjoint elements form anti-lattices. Thus, in a certain sense, the strength of the lattice structure of an operator algebra varies with the degree of commutativity of the algebra. (Received January 8, 1951.)

217. P. D. Lax: *Second order eliptic equations.* Preliminary report.

The alternative for the first boundary value problem is demonstrated: if $L[u]=0$ in D, $u=0$ in D, $u=0$ on B has n linearly independent solutions, then the inhomogenerous problem $L[u]=f$ in D, $u=\phi$ on B has a solution for at least one pair in any $(n+1)$-dimensional subspace of pairs (f, ϕ). L is any second order elliptic operator on an m-dimensional differentiable manifold with Hölder continuous coefficients; the boundary B of D is suitably restricted. The case where the coefficient of u in L is negative is handled first, by the method of superfunctions. For general L one writes $L=M+ku$ where M is of previous type; $u=T[w]$ is determined as the solution of

$M[u]=-kw+f$, $u=\phi$ on B. The solvability of $w=T[w]$ is studied in view of the complete continuity of T in the metric $\|u\|=\max|u|$, a byproduct of the method of superfunctions. These methods can also be used to construct singular solutions and solutions to nonlinear problems. (Received November 10, 1950.)

218. Benjamin Lepson: *Note on hyperdirichlet series with complex exponents.* Preliminary report.

Consider the series (1) $\sum_{n=0}^{\infty}P_n(z)e^{-\lambda_n z}$ and (2) $\sum_{n=0}^{\infty}A_n e^{-\lambda_n z}$, where $P_n(z)$ is a polynomial in the complex variable z of degree μ_n, A_n is the maximum of the moduli of the coefficients of $P_n(z)$, λ_n is complex, and $\lim_{n\to\infty}\mu_n/\lambda_n=0$. The set of points in the plane where (2) is absolutely convergent was shown by Ritt and Hille to be the interior plus a portion of the boundary of a convex domain D. It is shown here that the interior of the set of points where (1) is absolutely convergent is the domain D, and that (1) converges uniformly in any closed bounded set contained in D. Let E be the set of limit points of the roots of the $P_n(z)$, including those roots which appear infinitely often. Then the set of points exterior to D at which (1) converges absolutely is a subset of E of first category in the plane. This generalizes results of Valiron and of the author. (Received January 11, 1951.)

219t. L. B. Robinson: *A complete system of tensors.*

Having computed a complete system of semitensors attached to system (I) (Bull. Soc. Math. France (1940) p. 129), the author will compute a complete system of tensors where $r=2$. Set up a system of Riquier (Ω) whose solutions give the tensors. The tensors are obtained after (Ω) is solved directly. It is possible to obtain the same result by symbolic multiplication like that of Clebsch-Gordan. Let $r=1$. From a system of Riquier three solutions involving I are found, that is, the tensors $f_1\equiv\phi_1I_2-\phi_2I_1$, $f_2\equiv\psi_1I_2-\psi_2I_1$, $f_3\equiv\chi_1I_3$. The Greek letters are functions not containing I. For $r=2$, $f_1\times f_2\equiv\phi_1\psi_1I_{22}-(\phi_2\psi_1+\phi_1\psi_2)I_{12}+\phi_2\psi_2I_{11}$, $f_1\times f_1\equiv\phi_1^2I_{22}-2\phi_1\phi_2I_{12}+\phi_2^2I_{11}$, $f_2\times f_2\equiv\psi_1^2I_{22}-2\psi_1\psi_2I_{12}+\psi_2^2I_{11}$, $f_1\times f_3\equiv(\phi_1I_{23}-\phi_2I_{13})\chi_1$, $f_2\times f_3\equiv(\psi_1I_{23}-\psi_2I_{13})\chi_1$, $f_3\times f_3\equiv\chi_1^2I_{33}$. U is the aggregate of covariants $f_i\times f_j=\Phi_{ij}(U)$, Φ are arbitrary functions. Solve for I_{ij}. This gives a complete system of tensors. (Received January 29, 1951.)

220t. L. B. Robinson: *On a functional integral equation with a bearing on the group theory.*

Consider the equation $u(x,y)=f(x,y)+\lambda\int_1^{x^2}\int_1^{y^2}(1/(1+x_1^2))(1/(1+y_1^2))N(x,y;x_1,y_1)\cdot u(x_1,y_1)dy_1dx_1$. x, y are any points in the complex plane. $f(x,y)$, $N(x,y;x_1,y_1)$ are finite, even at infinity. The author, following the method of Fredholm and also making use of a cyclic and a permutation group, solves the above equation. (Received January 11, 1951.)

221t. Walter Rudin: *Green's second identity for generalized Laplacians.*

Let P denote a point in the plane domain D. Put $\Lambda F(P)=\lim_{r\to 0}4(m(F;P,r)-F(P))/r^2$, where $m(F;P,r)$ is the mean of the function F on the circle of radius r about P. Λ^*F, Λ_*F are defined likewise, with lim sup, lim inf in place of lim. Theorem: Suppose (1) U is continuous in D; (2) $\Lambda^*U(P)>-\infty$, $\Lambda_*U(P)<+\infty$ except possibly on a closed set of capacity zero; (3) there exists a function $y\in L$ on every compact subset of D, such that $y(P)\leq\Lambda^*U(P)$ in D; (4) V also satisfies (1), (2), (3), and

$V(P)=0$ outside a compact subset K of D. Then $\iint_D U(P)\Lambda V(P)dP=\iint_D V(P) \cdot \Lambda U(P)dP$. The proof is based on an earlier result concerning generalized Laplacians (Trans. Amer. Math. Soc. vol. 68 (1950) p. 279). (Received January 9, 1951.)

222t. Walter Rudin: *Positive infinities of potentials.*

Let R denote Euclidean n-space ($n \geqq 2$). Put $g(r)=r^{2-n}$ ($n>2$), $g(r)=-\log r$ ($n=2$). Theorem: Let E be a closed set of capacity zero in R, let G be an open set containing E. Then there exists a non-negative function f, summable on R, such that the superharmonic function (that is, the potential) $F(M)=\int_R g(MP)f(P)dP$ is infinite at every point of E, is continuous in $R-E$, and is harmonic in $R-\overline{G}$. If this result is compared with a theorem of Evans (Monatshefte für Mathematik und Physik vol. 43 (1936) p. 421), the main difference is that the potential obtained here is that of an absolutely continuous mass distribution. (Received January 9, 1951.)

223. James Sanders: *A class of partial differential equations of the fourth order.*

The equation (1) $L(u)=\sigma_1^2(x)u_{xx}+(\sigma_1(x)\sigma_2(x)\tau_1(y)/\tau_2(y))u_{yy}+\sigma_1\sigma_1' u_x+\sigma_1\sigma_2\tau_1 \cdot (1/\tau_2)'u_y=0$ has been considered by Bers and Gelbart (Trans. Amer. Math. Soc. vol. 56 (1944)). In an analogous manner this paper considers the equation (2) $L'[L(u)]=0$, where L' is derived from L by replacing σ_1 by $1/\sigma_2$ and σ_2 by $1/\sigma_1$. This equation generalizes the biharmonic equation and occurs in mathematical physics. It is shown that every solution of (2) is of the form (3) $u(x, y)=(\int_{x_0}^{x}dx/\sigma_1(x))P_1(x, y)+P_2(x, y)$, where $P_1(x, y)$ and $P_2(x, y)$ are solutions of (1). Conversely, every function of the form (3) is a solution of (2). Next, it is shown that every solution $u\in C^{(4)}$ of (2) in a domain D can be expanded in the neighborhood of every $z_0\in D$ in the form $u(x,y) =\sum_{n=0}^{\infty}\sum_{i=0}^{3}a_{ni}u_{ni}(x, y)$ where the a_{ni} are uniquely determined by u and explicit expressions for the u_{ni} are given. Finally, the solution of the boundary value problem $L'L(u)=0$, $u=f(s)$, $\partial u/\partial n=g(s)$, on the boundary C of D, is shown to be unique. (Received November 13, 1950.)

224. F. M. Stewart: *Uniqueness theorems for the equation $y' =f(x, y)$.*

By considering Lipschitz conditions and their generalizations on certain cones, new criteria for the uniqueness of solutions of $y'=f(x, y)$ are established. One somewhat special, but useful result is: If (i) $\mathfrak{M}$ is the set of all (x, y) at which the Lipschitz constant of f is infinite; (ii) $L(\rho)$ is the infimum of the Lipschitz constants of f at points whose distance from $\mathfrak{M}$ is at least ρ; (iii) $\int_{0+}^{a}L(\rho)d\rho$ converges; (iv) for each (x_0, y_0) neither of the half-lines $x=x_0\pm\theta$, $y=y_0\pm\theta f(x_0, y_0)$, $\theta\geqq 0$, is a half-tangent to $\mathfrak{M}$; then solutions of $y'=f(x, y)$ are unique. This result applies, as previously published theorems do not, to equations like $y'=g(x)+|y|^{\alpha}$ with $g(x)>0$, $0<\alpha<1$. (Received December 26, 1950.)

225t. Ferran Sunyer i Balaguer: *Values of entire functions represented by gap Dirichlet series.*

In two former notes (C. R. Acad. Sci. Paris vol. 224 (1947) pp. 1609–1610 and vol. 225 (1947) pp. 21–23) the author pointed out that if the entire function $F(z)$ is represented by a Taylor series which verifies certain gap conditions, the zeros of $F(z)-f(z)$ cannot be exceptional with respect to the proximate order of $F(z)$ by any meromorphic function $f(z)\not\equiv\infty$ of lower order. In the present paper the proof is given that an entire function, represented by a Dirichlet series verifying certain gap conditions, has no exceptional value, with respect to the proximate order (Ritt) of the function, in any strip of width greater than a quantity which depends only on the order (R) of the function. (Received January 9, 1951.)

226. H. F. Weinberger: *The connection between one-dimensional eigenvalue problems with interior boundary conditions.*

By an application of Green's theorem, a method is found for computing the eigenvalues and eigenfunctions of a regular self-adjoint differential problem of the form $p_n(x)u^{(n)}+\cdots+p_1(x)u'+p_0(x)u+\lambda u=0$ with boundary conditions at a finite number of points in terms of the Green's function and eigenfunctions of another problem with the same differential equation and any other self-adjoint boundary conditions at these points. The eigenvalues are determined by the behavior of a determinant of the Green's function and its derivatives at the "boundary points." The corresponding eigenfunctions are linear combinations of the eigenfunctions and of the Green's function and its derivatives. These results are an extension of a paper (Weinberger and Weinstein, *On the connection between the eigenvalues and eigenfunctions of some Sturm-Liouville problems*, Bull. Amer. Math. Soc. Abstract 57-2-138. (Received January 12, 1951.)

227. Albert Wilansky: *On norms of matrix type for* (c) *and* (m).

We redefine $\|x\| = \sup_n |\sum_k a_{nk}x_k|$ for $x \in (c)$ or $x \in (m)$, and inquire conditions of the matrix A that (c) and (m) be complete with the new norms. A known result (Toeplitz) implies that $\|A\| = \sup_n \sum_n |a_{nk}| < \infty$ is necessary. It is sufficient but not necessary that $\|A^{-1}\| < \infty$. If in addition $\lim_n a_{nk}^{-1}$ (element of matrix A^{-1}) exists for each k, the condition is also necessary; A will then be conservative. If (m) is complete, so is (c). The techniques used are the Banach-Steinhaus theorem and summability theory of the author, see Trans. Amer. Math. Soc. vol. 67, Bull. Amer. Math. Soc. vol. 55. (Received January 12, 1951.)

Applied Mathematics

228. René De Vogelaere: *A new family of periodic orbits in the cosmic rays problem: horseshoe orbits.*

The study of the motion of primary cosmic rays leads to a reversible Lagrangian system with two degrees of freedom. The three families first discovered are the equatorial family, the principal one, and the family of ovals; they were investigated mainly by Störmer, Lemaître, Vallarta, Lifschitz, and De Vogelaere. Here is a fourth family studied, two orbits of which were calculated by Störmer. A first approximation of the starting points is obtained by interpolation of the results of the orbits of Störmer and the supposed terminations of the family, the differential equations being solved by numerical integration. Most of the orbits have the form of a horseshoe and, on both sides, the family terminates on a line which corresponds to the equator. The stability and instability of the orbits are determined as well as the form of other periodic orbits near those of the family that have a zero characteristic exponent. (Received January 10, 1951.)

229. A. D. Fialkow and Irving Gerst: *The transfer function of a three terminal network.*

The transfer function of a three terminal (grounded) network is the ratio of the voltage between the output terminal and ground to the voltage between the input terminal and ground. The necessary and sufficient conditions that a real rational function $A(p)$ given by $A(p)=KN/D=K(p^n+a_1p^{n-1}+\cdots+a_n)/(p^m+b_1p^{m-1}+\cdots+b_m)$ be the transfer function of a passive three terminal network containing resistance and capacitance only are: (1) The roots of D are distinct negative numbers. (2) The roots of N may not be positive real but are otherwise arbitrary. (3) $m\geqq n$. (4) The number K satisfies the inequalities $0\leqq K<K_0$ where K_0 is the least of the three quantities K_d, b_m/a_n, 1 if $m=n$ and of the first two quantities if $m>n$. If $K_0\neq K_d$, then K may equal K_0. Here K_d is the least positive value of K (if it exists) for which the equation (*) $D-KN=0$ has a positive double root. (It follows that K_d is a certain root of the equation in K obtained by equating the discriminant of (*) to zero.) An algorithm for the synthesis of the network corresponding to $A(p)$ is given. It is possible to take account of any resistive source and load. (Received November 20, 1950.)

230t. A. D. Fialkow and Irving Gerst: *The transfer function of a four terminal network.*

The transfer function of a four terminal network is the ratio of the voltage at the output terminals to the voltage at the input terminals. The necessary and sufficient conditions that a real rational function $A(p)$ be the transfer function of a passive four terminal network containing resistance and capacitance only are identical with those stated in the preceding abstract except that condition (2) is lacking (that is, the roots of N are arbitrary) and condition (4) must be modified to read: (4′) The number K satisfies the inequalities $-K_0<K<K_0$ where K_0 is the least of the three quantities $|K_d|$, $|b_m/a_n|$, 1 if $m=n$, and of the first two quantities if $m>n$. If $K_0\neq|K_d|$, then K may actually equal $\pm K_0$. Here K_d is that real value of K of smallest absolute value (if it exists) for which the equation $D-KN=0$ has a positive double root. A synthesis procedure is given. Analogous results obtain for networks which contain resistance and inductance only or inductance and capacitance only. (Received November 20, 1950.)

231. G. E. Forsythe and T. S. Motzkin: *Asymptotic properties of the optimum gradient method.*

To minimize a well-behaved real-valued function F of an n-dimensional vector x, start with x_0 sufficiently near to a minimum x^*, and successively determine the $x_{k+1}=x_k-\alpha$ grad $F(x_k)$ which minimizes $F[x_{k+1}(\alpha)]$ (Cauchy, 1847; see H. B. Curry, *The method of steepest descent for non-linear minimization problems*, Quarterly of Applied Mathematics vol. 2 (1944) pp. 258–261). Commonly x_k will ultimately behave as though $F(x)=|Ax-b|^2$, where A and b are an appropriate matrix and vector. For $n=3$ the authors prove that x_k-x^* is asymptotically a linear combination of the eigenvectors belonging to the largest and smallest eigenvalue of A^TA (A^T = transpose of A). Furthermore $v_{2k}=\text{grad } F(x_{2k})/|\text{grad } F(x_{2k})|$ has a limit v^*, and $|v_{2k}-v^*|\to 0$ like some q^k. (Received January 8, 1951.)

232. Jenny E. Rosenthal: *Inapplicability of standard techniques of conformal mapping to certain types of physical problems.*

The method of conformal mapping to determine the potential distribution near the edge of a parallel plate condenser, first used by Helmholtz in 1868, consists of the transformation of two infinite lines into two semi-infinite ones. It has since been given in numerous texts on applied mathematics and mathematical physics. It is shown in the present paper that the Helmholtz "solution" is not unique. Infinitely many functions will transform two infinite lines into two semi-infinite ones. Various classes of such functions are constructed. It is shown that the basic difficulty is due to the fact that while all such "solutions" satisfy the boundary conditions on the condenser plates, none of them gives a physically reasonable potential distribution at infinity. By means of a standard successive transformation it is also shown that the Helmholtz transformation gives spirals as equipotential lines in the case of two intersecting portions of straight lines raised to different potentials, a manifest impossibility physically. This standard transformation is a simple exponential known to transform two parallel infinite lines into two semi-infinite lines diverging from a vertex. (Received January 11, 1951.)

Geometry

233. Iacopo Barsotti: *Local properties of algebraic correspondences.*

Let D be an irreducible algebraic correspondence between the irreducible varieties F, V; if G is an irreducible subvariety of F, let D' be the algebraic correspondence between G and V obtained by "reducing" D. The multiplicity with which a component D^* (having the right dimension) of D' has to be "counted" is shown to be equal to the ratio between the multiplicities of the local rings of D^* and G with respect to any set of parameters of the latter. Other expressions are obtained for this multiplicity, in terms of the ramification of certain valuations. These expressions remain valid when G is fundamental for D, provided that F is analytically irreducible at G. Let W be an irreducible subvariety of V; it is proved (under certain restrictions) that D can be "reduced" to a correspondence between G and W by reducing D' from V to W, and that the result is the same if the two reductions are performed in the opposite order. The method used is algebro-analytical, that is, completion of local rings and valuations; a short proof of the associativity formula for geometric domains is supplied. The results have immediate application (not given in the paper) to the theory of intersection of cycles. The paper closes with a jacobian criterion for multiplicity one (for multiplicity p^m in the language of Weil). (Received January 5, 1951.)

234*t*. Stefan Bergman. *On models of four-dimensional domains.*

It is useful (as indicated in Amer. Math. Monthly vol. 53 (1946) p. 20) to employ in teaching and research models of domains of four-dimensional space which arise in the theory of functions of two complex variables. The procedures for constructing different geometrical configurations and performing geometrical operations can be represented analytically by certain formulas, for example, various geometrical operations (rotation, stretching, and so on) are equivalent to linear transformations, $X_k = \sum_{\nu=1}^{4} a_{k\nu} x_\nu$, $k = 1, 2, 3, 4$. Evaluation of the above formulas for a great number of points, needed in these procedures, can be performed conveniently by using punch-card machines or other computing devices. The author indicates procedures for the performance of these operations and the preparation of the stills by the use of automatic computing devices such as punch-cards or a differential analyzer. (Received February 7, 1951.)

Statistics and Probability

235. W. R. Wasow: *On the duration of random walks.*

In a bounded domain B of n-dimensional Euclidean space E consider a discrete random walk that begins at a point x of B. The nature of the random walk is characterized by a transition probability function $F(y, x)$. The duration N_x of the walk, that is, the number of steps before the moving particle leaves B for the first time, is shown to possess, under mild hypotheses, a moment generating function $\phi(s, x)$ that is the unique solution of the integral equation $\int_B \phi(s, y) d_a F(y, x) - e^{-s}\phi(s, x) = \int_{E-B} d_y F(y, x)$. If $F(y, x)$ depends on a parameter μ in such a way that the transition probability is increasingly concentrated around x as $\mu \rightarrow 0$, then $\phi(s, y)$ can, under certain assumptions, be asymptotically approximated by the solution of a boundary value problem for an elliptic differential equation. This equation involves only the first and second moments of $F(y, x)$. It is further shown that the distribution function of μN_x tends, as $\mu \rightarrow 0$, to a solution of a related parabolic differential equation problem. As an application, asymptotic expressions for the variance of N_x in special cases and estimates for the likelihood of very long walks are obtained. (Received December 26, 1950.)

Topology

236. R. D. Anderson: *Hereditarily indecomposable plane continua.*

The author demonstrates the existence of a hereditarily indecomposable plane continuum, not separating the plane, which is not homeomorphic to a pseudo-arc (that is, a chained hereditarily indecomposable plane continuum). There are, in fact, uncountably many distinct such continua including one which contains no pseudo-arc and another all of whose proper subcontinua are pseudo-arcs. As a corollary of the above, it follows that there exist uncountably many mutually exclusive continua in the plane such that no one separates the plane and no one is chainable. (Received January 10, 1951.)

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