THE APRIL MEETING IN STANFORD

The four hundred seventieth meeting of the American Mathematical Society was held at Stanford University, Palo Alto, California, on Saturday, April 28, 1951. Approximately 127 persons attended, including the following 91 members of the Society:

H. L. Alder, T. M. Apostol, Richard Arens, H. A. Arnold, H. M. Bacon, W. G. Bade, G. A. Baker, Clifford Bell, R. E. Bellman, Benjamin Bernholz, Z. W. Birnbaum, David Blackwell, W. W. Bledsoe, F. H. Brownell, H. D. Brunk, C. L. Clark, J. G. van der Corput, Richard Courant, A. C. Davis, E. A. Davis, R. Y. Dean, S. P. Diliberto, R. P. Dilworth, Brother V. Dominic, Melvin Dresher, Roy Dubisch, L. K. Durst, Arthur Erdélyi, D. J. Ewy, Harley Flanders, G. E. Forsythe, R. A. Fuchs, C. M. Fulton, P. R. Garabedian, M. A. Girshick, G. E. Gourrich, G. J. Haltiner, J. G. Hardy, C. A. Hayes, G. B. Hedrick, J. G. Herriot, M. R. Hestenes, Edwin Hewitt, J. L. Hodges, C. G. Jaeger, Fritz John, R. M. Lakness, Cornelius Lanczos, G. E. Latta, D. H. Lehmer, R. B. Leipnik, Hans Lewy, J. C. C. McKinsey, A. V. Martin, A. B. Mewborn, W. E. Milne, C. B. Morrey, F. R. Morris, T. S. Motzkin, R. G. Needels, Ivan Niven, I. L. Novak, H. A. Osborn, Edmund Pinney, George Pólya, R. M. Redheffer, M. M. Resnikoff, J. B. Robinson, R. M. Robinson, E. B. Roessler, P. G. Rooney, Herman Rubin, Seymour Sherman, M. L. Stein, Robert Steinberg, R. G. Stoneham, E. G. Straus, A. C. Sugar, M. V. Sunseri, Irving Sussman, Gabor Szegö, R. B. Talmadge, Alfred Tarski, M. A. Weber, J. W. Weihe, F. J. Weyl, P. A. White, A. L. Whiteman, A. R. Williams, G. M. Wing, František Wolf.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Arthur Erdélyi of the California Institute of Technology delivered at 11:00 A.M. an address entitled *The analytic theory of systems of partial differential equations.* Professor Erdélyi was introduced by Professor George Pólya. There were sessions for contributed papers in the morning and afternoon, presided over by Professors Gabor Szegö, R. E. Bellman, and W. E. Milne. Following the meetings, those attending were guests at tea in the Stanford Union.

Following are abstracts of papers presented at the meeting. Papers whose abstracts numbers are followed by "*t*" were presented by title. Paper number 380 was presented by Professor Hyers, number 389 by Professor Hewitt, number 393 by Professor Birnbaum, and number 396 by Professor Straus.

ALGEBRA AND THEORY OF NUMBERS

369*t*. H. W. Becker: *Euler numbers in combinatory analysis.*

The Euler-Andre numbers (Netto, *Combinatorik*, 1927, pp. 105–112) have an interpretation; $2A_n$ is the number of alternating permutations of n letters. Their generating function is $e^{tA} = \sec t + \tan t$, from which $A_{n+1} = (A + jE)^n$, $j = (-1)^{1/2}$, $E_{2m-1} = 0$, $E_{2m} = (-)^m A_{2m}$. A new interpretation is: $A_{n+1} = W_n$ is the number of n letter words such that the mth letter of the alphabet may not occur prior to the mth position in the word, and no letter occurs more than twice. Their generating function is $e^{tW} = (\sec t)e^{tA}$, from which $W_{n+1} = (W+A)^n = (W+S)^{n+1}$, $S_m = \sin \pi m/2$. The dictionary of these words may be classified the usual ways, according to terminal letter, maximum letter, number of different letters, number of singletons, position of last "a," and so on, thus yielding new recurrences and breakdowns of the Euler numbers. If these words are restricted to being rhyme schemes $H_n \subset W_n$, then H_n (also the number of self-conjugate permutations of n letters) is an Hermite number, with generating function $e^{tH} = \exp (t+t^2/2)$. If further restricted to consecutive rhyme, $F_n \subset H_n$, then F_n (isomorphically the number of compositions of n into 1's and 2's) is a Fibonacci number, with generating function $1/(1-tF) = 1/(1-t-t^2)$. But it is an unsolved problem to find a generating function for $n! \supset W_n$, the first two terms of which are constituted by e^{tW}. (Received March 12, 1951.)

370*t*. H. W. Becker: *The closed diagonal binomial coefficient expansion.*

The title refers to sums $\|U+V\|^n = \sum U_{n-m} \cdot V_m \cdot \|n, m\|$, the notation emphasizing that for given n the successive $\|n, m\| = (n-m, m)$ are found on the diagonal of the table of binomial coefficients, ascending from the left. Four different combinatory interpretations are given, of the Fibonacci series $F_n = \|1+1\|^n$, and of the Hermite series $H_n = \|1+(n)/2\|^n$, (n) being here the umbra of a Jordan factorial. Application is made to various configurations of Lucas, *Theorie des nombres*, 1891, pp. 218–219. If $N_n = (2n)!/n!(n+1)!$, which incidentally has more than forty different combinatory interpretations, then the umbral product $N^{k*}\|N-1\|^n = 0$, 1, or $(n+1)(n+2k)_{k-1}/k!$ according as $n \geqq -2k$, $k=0$, or $k>0$. There are such expansions of zero and fractions as $\|1-1\|^{n-1} = (2/3^{1/2}) \sin (\pi n/3)$, $\|1-1/2\|^{4n-1} = \|1-1/3\|^{6n-1} = 0$, $\|1-1/4\|^n = (n+1)/2^n$, and so on, suggested by a problem of Coxeter, Amer. Math. Monthly vol. 52 (1945) pp. 100–101. (Received March 12, 1951.)

371*t*. P. W. Carruth: *Sums and products of ordered systems.*

A non-empty set R of elements in which a reflexive binary relation $r \geqq r'$ is defined is termed an *ordered system*. Let R be an ordered system, and for each element r in R, let S_r be an ordered system. Then an ordered or lexicographic product and an ordered sum over R of the systems S_r have been defined (see, for example, Day, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 1–43). Necessary and sufficient conditions for an ordered product to be a lattice, complete lattice, chain, cardinal, and so on, respectively are obtained. Analogous results concerning ordered sums also are given. In the proofs use is made of results of Day, loc. cit. (Received February 28, 1951.)

372. L. K. Durst: *Apparition of primes in elliptic sequences (equianharmonic case).*

Let E be the ring of quadratic integers $a+b\rho$, $\rho^3=1$. If $\omega_2/\omega_1=\rho$, $g_2=0$, then $\psi_\mu(u) = \sigma(\mu u)/\sigma(u)^{N\mu}$ generates a mapping of E into the ring $E[\wp(u), g_3]$ which preserves division. An explicit arithmetical restriction is given for the apparition of prime ideals in this divisibility mapping, solving for this case a problem proposed by Morgan Ward (Amer. J. Math. vol. 70 (1948) p. 74). (Received March 15, 1951.)

373. Ivan Niven: *Functions which represent prime numbers.*

Given any real number $c>8/3$, it is proved that there exists a real number A

such that $[A^{c^n}]$ is a prime for every positive integer n; also for any given $A>1$ there exists a real number c yielding the same conclusion. The first part of this result was proved in the case $c=3$ by W. H. Mills [*A prime representing function*, Bull. Amer. Math. Soc. vol. 63 (1947) p. 604] and in the case of positive integers $c>3$ by L. Kuipers [*Prime-representing functions*, Neder. Akad. Wetensch. vol. 53 (1950) pp. 309–310]. The proof of the first part is a slight rearrangement of Mills' proof, and the proof of the second part employs the same basic idea in a different setting. (Received March 9, 1951.)

374*t*. Ernst Snapper: *Completely primary rings*. IV.

In the three papers *Completely primary rings*, I, II, III (Ann. of Math. (2) vol. 52 (1950), vol. 53 (1951)), no chain conditions were imposed on the rings. In the present paper a completely primary ring S whose ideals satisfy the two chain conditions is investigated. (Either chain condition implies the other.) It is shown that S always contains a *coefficient ring* R, that is, a completely primary ring which is a canonical extension of the prime ring of S and whose residue class field is equal to that of S. The first part of the paper is concerned with the properties of R. It is then observed that, since R is always a principal ideal ring and S is an R-module with a finite number of generators, one can introduce the elementary divisors of S, that is, the elementary divisors of the R-module S. The invariants of S which arise from this elementary divisor theory are discussed in the second part of the paper. In the last part of the paper these results are applied to the case when S is a principal ideal ring. (Received March 14, 1951.)

375*t*. Morgan Ward: *The diophantine equation $xy=zw$ in semi-groups*. Preliminary report.

Let I be a commutative semi-group with a unit partially ordered by the usual division relation. Assume that the diophantine equation $xy=zw$ always has solutions in I of the form $x=kl$, $y=mn$, $z=km$, $w=ln$; k, l, m, n in I. If in addition every element has only a finite number of nonequivalent divisors, I may be a group G, a distributive lattice L, or isomorphic to a subset M of the positive integers closed under multiplication. These are essentially the only possibilities; I is always isomorphic to a direct product $G\times L\times M$. (Received March 16, 1951.)

376. A. L. Whiteman: *Numbers of solutions of equations in finite fields*.

Let p be an odd prime and $GF(p^n)$ the Galois field of order p^n. Let g be the generator of the cyclic group formed by the nonzero elements of $GF(p^n)$ under multiplication. Let $m_1, m_2, \cdots, m_s$ be s divisors of p^n-1 and put $p^n-1=m_im_i'$, $i=1, 2, \cdots, s$. In a series of three papers (Proc. Nat. Acad. Sci. U. S. A. vol. 33 (1947) pp. 236–242; vol. 34 (1948) pp. 62–66; vol. 35 (1949) pp. 681–685) Vandiver considered the problem of finding the number of solutions $(j_1, j_2, \cdots, j_s)$ in $\gamma_1, \gamma_2, \cdots, \gamma_s$ of $1+g^{j_1+m_1\gamma_1}+g^{j_2+m_2\gamma_2}+\cdots+g^{j_s+m_s\gamma_s}=0$, for a fixed set of numbers, $j_1, j_2, \cdots, j_s$ with the j_i in the range $0, 1, \cdots, m_i-1$, and the γ_i in the range $0, 1, \cdots, m_i'-1$. Explicit expressions involving p and n were obtained in the case $s=2$ for the sums $\sum_{j_1,j_2,\ldots,j_s}(j_1, j_2, \cdots, j_s)(j_1+k_1, j_2+k_2, \cdots, j_s+k_s)$, where the j_i range independently over $0, 1, \cdots, m_i-1$, and $k_1, k_2, \cdots, k_s$ is a fixed set of nonnegative integers. In the present paper Vandiver's method is extended to derive similar results in the case where $s=3$ and in the general case where the m_i are prime each to each. (Received January 23, 1951.)

ANALYSIS

377. F. H. Brownell: *Translation invariant measure over separable Hilbert space.*

This paper investigates the existence of a translation invariant, nontrivial measure ϕ over the prototype real separable Hilbert space l_2, or more generally over the subset $X=\{x\in l_2 \mid |x_n| \leqq h(n)$ for all $n\}$, where $h(n)$ is a fixed function over the positive integers with values positive real or $+\infty$. Obviously $X=l_2$ if $h(n)\equiv+\infty$. Precisely the conditions on ϕ are that it should be defined non-negative and completely additive over the Borel class of subsets of X, that it be invariant over l_2 vector translations of subsets of X remaining inside X, and to be nontrivial that $\phi(X)>0$ and $\phi(A)<+\infty$ for some nonempty A open in X under some topology. The results found are that if, as when $X=l_2$, $\lim\inf_{n\to\infty} h(n)>0$, then no such ϕ exists for the norm metric topology, and hence none for the Cartesian product topology either. Also if $h(n)=+\infty$ for infinitely many n, no ϕ exists for the product topology. However if $\sum_{n=N}^{\infty} [h(n)]^2<+\infty$ for some finite N, the two topologies coincide, X is locally compact under either, and ϕ exists unique up to a constant factor. Extensions to other subsets of l_2 than X and to the case where $\lim\inf_{n\to\infty} h(n)=0$ but $\sum^{\infty} [h(n)]^2=+\infty$ are also considered. (Received March 16, 1951.)

378. H. D. Brunk: *Three-dimensional analogues of the cardinal series.* Preliminary report.

The cardinal series gives an entire function assuming given values at equally-spaced points on a line. Two three-dimensional analogues of the cardinal series have been studied, and a convergence property has been proved, generalizing to a strip region that of de la Vallee Poussin for the line (Académie Royale de Belgique. Bulletin de la Classe des Sciences vol. 1 (1908) pp. 319–410). (Received March 15, 1951.)

379. J. G. Herriot: *Partial-sum couplings for double Fourier series.*

Let $s_{mn}(x, y)$ be the rectangular partial sums of the double Fourier series of an integrable periodic function $f(x, y)$. It is well known that convergence of these partial sums is not a local property, even if the manner in which $m, n\to\infty$ is restricted by keeping m/n and n/m bounded. By forming suitable couplings of the partial sums similar to those studied by Rogosinski for single Fourier series (Math. Ann. vol. 95 (1926) pp. 110–134), there is obtained a method of summation for double Fourier series which possesses the localization property. Let $\kappa_{mn}(x, y, p, q, h, k) =(1/4)[s_{mn}(x+h, y+k)+(-1)^{p-1}s_{mn}(x+h+p\pi m^{-1}, y+k)+(-1)^{q-1}s_{mn}(x+h, y+k+q\pi n^{-1})+(-1)^{p+q}s_{mn}(x+h+p\pi m^{-1}, y+k+q\pi n^{-1})]$, where p and q are integers. If h_m and k_n are null sequences, then, at points of continuity of $f(x, y)$, $\kappa_{mn}(x, y, p, q, h_m, k_n)$ converges restrictedly to $(1/4)[1+(-1)^{p-1}][1+(-1)^{q-1}]f(x, y)$. For odd values of p and q this sequence exhibits a Gibbs' phenomenon at ordinary discontinuities of $f(x, y)$. If one forms a coupling of $\kappa_{mn}(x, y, p, q, h, k)$ by changing the signs of p, q and h, k, a method of summation is obtained which is effective at points where $f(x, y)$ satisfies more general conditions than continuity. The proofs are obtained by studying the kernels of the transformations. (Received March 12, 1951.)

380. D. H. Hyers and S. M. Ulam: *Approximately convex functions.*

A real-valued function $f(x)$ defined on a convex subset S of Euclidean space E_n will be called *ε-convex* if $f(hx+(1-h)y)\leqq hf(x)+(1-h)f(y)+\epsilon$ for all x and y in S and for $0\leqq h\leqq 1$. Here ϵ is a fixed positive number. The principal theorem is this. If

S is an open convex set (bounded or unbounded) in E_n and if $f(x)$ is ϵ-convex and continuous on S, then there exists a convex function $g(x)$ such that $|f(x)-g(x)| \leqq k\epsilon$, for all x in S, where $2k=1+(n-1)(n+2)/2(n+1)$. The proof is carried out first for polyhedral functions (whose graphs are polyhedral surfaces) which are ϵ-convex and the result is extended to continuous functions by a limiting process. (Received March 5, 1951.)

381. Fritz John: *Solution of parabolic equations by finite differences.*

A differential equation of the form (1) $u_t=F(x, t, u, u_x, u_{xx})$ $(-\infty<x<+\infty)$ with initial values $u(x, 0)=f(x)$ is replaced by the difference equation (2) $\nabla u =F(x, t, u, \Delta u, \Delta^2 u)$, where $\nabla u=[u(x, t+k)-u(x, t)]/k$, $\Delta u=[u(x+h, t)-u(x, t)]/h$, $\Delta^2 u=[u(x-h, t)-2u(x, t)+u(x+h, t)]/h^2$. The functions $f(x)$, $F(x, t, u, p, q)$ shall have continuous derivatives up to the fourth order, which are bounded for bounded t, u, p, q and all x. Moreover F_q shall be positive. It is proved that for given f, F there exists a $T>0$, such that for $0\leqq t\leqq T$ and for all sufficiently small h, k/h^2, the solution of (2) will converge towards a solution of (1) for $h\rightarrow 0$. If (1) is of the special form $u_t =a(x, t)u_{xx}+b(x, t)u_x+c(x, t, u)$ $(a>0)$, the solution of (2) converges for $h\rightarrow 0$ under the less restrictive assumptions that $f(x)$, $c(x, t, u)$ and its first derivatives, $a(x, t)$, $b(x, t)$ and their first and second derivatives are continuous and bounded for bounded t, u. (Received March 12, 1951.)

382. Cornelius Lanczos: *Integration by weighted sums.*

The customary definition of an indefinite integral by means of a sum of equidistant ordinates up to x has generally slow convergence. It is based on the difference equation $\Delta u/\Delta x=f(x)$, letting Δx converge to zero. In this paper the customary procedure is modified by associating with $u(x)$ a new $v(x)$ on the basis of the operational relation $du/dx=\Delta v/\Delta x$. After obtaining $v(x)$ by the customary summation process, one performs the transformation from $v(x)$ to $u(x)$. This transformation amounts to a weighting of the partial sums by pre-assigned weight factors. Numerical integration is thus changed into a simple summation process, augmented by an additional weighting which greatly increases the accuracy obtained. The process can be generalized to linear partial differential operators with constant coefficients. It leads to an increase of the mesh size used in changing the differential equation into a difference equation. Analytically interesting is the fact that sums which do not converge due to too great nearness of a point of infinity of $f(x)$ become convergent after the proper weighting and yield the indefinite integral at every point except the point of infinity. (Received March 9, 1951.)

383. R. B. Leipnik: *Solution of nonhomogeneous linear differential equations with infinitely differentiable coefficients.* Preliminary report.

A formal solution is found for a nonhomogeneous linear differential equation with infinitely differentiable coefficients as a sum of products, one factor of which depends only on the coefficients and their derivatives, the other on the right-hand side only. Convergence is studied. (Received March 14, 1951.)

384. Hans Lewy: *A note on harmonic functions and a hydrodynamical application.*

Let $U(x, y)$ and $V(x, y)$ be conjugate harmonics near the origin for $y<0$ and assume U, V, $\partial U/\partial x$ continuous for $y\leqq 0$. If, for $y=0$, $\partial U/\partial y=A(x, U, V, \partial U/\partial x)$ with A analytic at the argument values occurring on $y=0$, then U and V are analytically extensible across $y=0$. This theorem is proved and applied to the irrotational motion of a homogeneous incompressible liquid subject to gravity. It is shown that in a steady motion relative to a Galilei system the free surface, where the pressure is constant, is necessarily analytic, provided the particle occupying the free surface is not at rest. (Received March 12, 1951.)

385. T. S. Motzkin: *Periodicity regions of sequences with a piecewise linear recursion formula.* Preliminary report.

The iteration of piecewise linear transformations of d-space (or of piecewise analytic transformations of a d-dimensional variety), important i.a. in the theory of the approximative solution of linear (or general) equations and inequalities, may lead to a periodicity behavior essentially different from the usual one (Poincaré, Weyl, F. John) where periodicity either holds for every starting-point or at most for a set of points with measure 0. Already $a_{n+1}=a_n+|a_n|$ (or $(a_n+|a_n|)\exp(-a_n^{-2})$ with piecewise analyticity and differentiability of all orders), $d=1$, has a d-dimensional region of periodicity and of aperiodicity. A more intricate situation holds for $a_{n+3}+a_n = \max(a_{n+1}, a_{n+2})$, $d=3$. To decide on periodicity and to find the length p of the period, proceed until $a_{k-1}\leqq a_k \geqq a_{k+1}$, $a_k\geqq 0$, which will occur for some $k=1, \cdots, 9$, and let $r=(2a_k-a_{k-1}-a_{k+1})/(a_{k-1}+a_{k+1}-a_k)$. Then if r is irrational and >0, there is no period; if $r=0/0$, $p=1$; if $r=-2$, $p=2$; if $r<-2$, $r=\infty$ or $r=0$, $p=8$; if $r=1$, $p=9$; if r or $1/r=1+s/t$ (s and t coprime and >0), $p=8s+18t$. Every even number, with 36 exceptions (4, 6, 10, $\cdots$, 136, 180, 228), can be a period. The sequence is always bounded and attains its upper bound infinitely many times. Some of these results can be generalized to the sequence $a_{n+d}+a_n=\max(a_{n+1}, \cdots, a_{n+d-1})$. (Received April 24, 1951.)

386. M. L. Stein: *On methods for obtaining solutions of fixed end point problems in the calculus of variations.*

Two methods are proposed for constructing solutions to the problem of minimizing an integral in a certain class of arcs joining a pair of fixed points. For both methods an initial estimate of the solution is made. From this estimate a new one is obtained in a definite way, and so forth. One of the proposed procedures is a generalization of Newton's method. It is shown to converge to a solution of the Euler equations associated with the integral to be minimized and (under the proper conditions) to yield a strong relative minimum. The second of the proposed procedures is a "gradient" method. It is shown to be particularly applicable to an integral whose integrand is quadratic. Conditions for convergence to a strong relative minimum are given. (Received March 4, 1951.)

387. F. G. Tricomi: *A new entire function related to a well known noncontinuable power series.*

In the study of the statistical distribution of certain bacteria mutants there arises (see Bull. Amer. Math. Soc. Abstract 57-4-394) a new entire function $G(t) = \sum_{n=1}^{\infty} t^n/[(a^n-1)n!]$ in the case $a=2$. This function is related to the well known noncontinuable power series $\sum_{n=0}^{\infty} z^{a^n}$. Precisely, if one puts $z=e^t$, $F_m(t) = \sum_{n=0}^{m-1}\exp(a^n t)$, then $F_m(t)=m+G(a^m t)-G(t)$. Among other things, the function G is connected to the quite strange doubly periodic function $P(u)=\sum_{m=-\infty}^{\infty} f(u+m)$ where $f(t)=\exp(t/A-e^{t/A})$, $A=1/\log a$, a function which exists in only half of each period parallelogram. Furthermore this function is *almost constant* on the real axis as long as log a is not too large. For instance in the case $a=2$, one has $|P(x)-A| < 0.0000143$. This inequality can be deduced from the elegant Fourier expansion of the function $P(u)=A\sum_{n=-\infty}^{\infty}\Gamma(1-2nA\pi i)\exp(2nu\pi i)$. Moreover the functions $G(u)$ and $P(u)$ are related also to the *principal solution* of the difference equation $\Delta\phi(u)=f(u)$, where f has the previous expression. Actually the function $G(t)$ is an entire function of order *one* which attains any finite value in an infinity of points of the complex t-plane. On the real axis $G(t)$ is *totally monotone* in the sense that all its derivatives are positive there. But, while $G(t)$ grows like $e^{t/a}$ for $t>0$, $-G(t)$ grows logarithmically as $t\rightarrow -\infty$. This fact is connected with the extreme "skewness" of the frequency curve of the bacteria mutants. (Received March 12, 1951.)

388. František Wolf: *Analytic perturbations of operators.*

If (i) $A(\lambda)=\sum_0^\infty A_k\lambda^k$, for $|\lambda|<\delta$, a bounded operator in Hilbert space is normal for $\lambda=\lambda_n$, $\lim \lambda_n=0$, (ii) μ_0 is an isolated point of $\sigma(A_0)$, the spectrum of A_0, (iii) $A_1-\mu_1 I$ is completely continuous, then there is a sequence of analytic functions $\mu_k(\lambda)$ and a real function $\epsilon(\lambda)$ such that $\lim \epsilon(\lambda)=0$ and $\mu\in\sigma(A(\lambda))$ in a certain neighborhood of μ_0 implies either $|\mu-\mu_0-\mu_1\lambda|<\lambda\in(\lambda)$ or μ equal to one of the numbers $\mu_k(\lambda)$. The domains of analyticity D_k of $\mu_k(\lambda)$ might be such that the intersection of the D_k's may be the single point $\lambda=0$. If A_1 does not have μ_1 as an eigenvalue, then the conclusion is "then there exists a sequence of analytic functions $\mu_k(\lambda)$ such that $\mu\in\sigma(A(\lambda))$ near μ_0 implies $\mu=\mu_k(\lambda)$ for some k." This theorem is a generalization of a theorem of F. Rellich (Math. Ann. vol. 113 (1936) pp. 600–619) who restricts the multiplicity of μ_0 as an eigenvalue of A_0 to be finite, but omits condition (iii). He shows that without any additional condition the finiteness of the multiplicity is essential. The author was able to deduce the theorem from a result which covers the asymptotic behavior of $\sigma(A(\lambda))$ near μ_0 for sufficiently small λ in the case of bounded operators in Banach space (cf. abstract at the International Congress of Mathematicians, Cambridge, Mass., 1950). (Received March 25, 1951.)

389. Kosaku Yosida and Edwin Hewitt: *Finitely additive measures.*

Let $\mathfrak{M}$ be an algebra of subsets of a set T. A real-valued finitely additive measure ϕ defined on $\mathfrak{M}$ is said to be purely finitely additive if for every countably additive measure γ such that $0\leqq\gamma\leqq\phi^+$ or $0\leqq\gamma\leqq\phi^-$, one has $\gamma=0$. It is proved that every finitely additive measure ϕ defined on $\mathfrak{M}$ admits a unique resolution into a purely finitely additive part and a countably additive part. If $\mathfrak{M}$ is a σ-algebra and a certain σ-ideal $\mathfrak{N}\subset\mathfrak{M}$ is designated, then the set of all finitely additive real-valued measures which vanish on $\mathfrak{N}$ is a representation of the conjugate space to the Banach space $\mathfrak{L}_\infty$ defined for $T, \mathfrak{M}$, and $\mathfrak{N}$. A number of peculiar finitely additive measures are constructed, using this fact, for the case $T=R$, $\mathfrak{M}=$Lebesgue measurable sets, $\mathfrak{N}=$set of Lebesgue measure 0. It is also shown that for general T, $\mathfrak{M}$, and $\mathfrak{N}$, all finitely additive measures have counterparts which are countably additive Borel measures on a suitable compact Hausdorff space. The properties of pure finite additivity and countable additivity are identified in terms of the behavior of these countably additive Borel measures. (Received March 12, 1951.)

Applied Mathematics

390. E. A. Davis: *A dynamic economic model.* Preliminary report.

This paper is concerned with the construction of a mathematical model for an economic system. It uses techniques developed by Griffith C. Evans (Econometrica vol. 2 (1934) pp. 37–50), and F. W. Dresch (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 134–141). The model deals with operation of industries over an extended time interval. Each industry is considered to produce either a capital good, a consumer good, or a direct factor of production. It is presumed that capital and consumer good industries fix production rates so as to maximize integrals that define profits under the assumption that output is sold at the rate at which it is produced. In the maximization, input rates for productive factors and period of production are supposed subject to enterpriser control. The existence of non-constant inventory stocks is considered. A banking system is admitted and some investigation of monetary flows is made. An expression for the velocity of circulation of money results. Finally, index quantities (including indices for "period of production") are defined and relations existing among the index quantities are found to be generalizations of fundamental relations holding in a three industry model of the type considered. (Received March 15, 1951.)

391. G. E. Forsythe: *Generation and testing of* 1,217,370 *"random" binary digits on the SWAC.* Preliminary report.

Let $f_i(k)=0$ or 1 $(k=1, \cdots, n_i; i=1, \cdots, r)$, where $n_1, \cdots, n_r$ are relatively prime. Let each $f_i(k)$ be extended periodically with period n_i over the domain $k=1, 2, \cdots$. The sequence $F_k=\sum_{i=1}^r f_i(k)$ (modulo 2) $(k=1, 2, \cdots)$ has period at most $N=n_1n_2\cdots n_r$. J. B. Rosser (unpublished) proposed $F_1, F_2, \cdots, F_{\theta N}$ $(\theta \ll 1)$ as possible "random" digits for sampling problems. Let $n=n_1+\cdots+n_r$. Let $\mathfrak{f}$ $[\mathfrak{F}]$ be the space of all n-uples f [all N-uples F] of independent binary digits. With each f one may associate functions $\{f_i(k)\}$, and hence may determine a unique $F(f)$ by Rosser's process. Let $Z(F)$ be the number of 1's in F. For $m=0, 1, 2, 3$ the present author shows that the mth moment of $Z(F)$ in $\mathfrak{F}$ equals the mth moment of $Z[F(f)]$ in $\mathfrak{f}$. For $r=4$, $n_1=35$, $n_2=34$, $n_3=33$, $n_4=31$, $N=1,217,370$, and for one choice of f, the corresponding N-uple $F(f)$ was generated on the National Bureau of Standards Western Automatic Computer (SWAC). Also on the SWAC, all sums s_j of 100 consecutive digits F_k were formed, and distributions were made of the s_j in twelve disjoint groups of 1000 consecutive sums s_j. The χ^2 test showed 11 of the 12 distributions to fit reasonably well the theoretical (binomial) distribution of 1000 s_j. The twelfth was a bad fit: $\chi^2=47.24$ for 23 degrees of freedom $(P \ll .01)$. (Received March 12, 1951.)

392*t*. G. E. Forsythe and T. S. Motzkin: *Acceleration of the optimum gradient method.* Preliminary report.

In Cauchy's optimum gradient method for determining the solution x^* of the system $Ax=b(|A|\neq 0)$ of n linear equations in n unknowns, one lets $F(x)=|Ax-b|^2$ and successively determines the $x_{k+1}=x_k-\alpha$ grad $F(x_k)$ which minimizes $F[x_{k+1}(\alpha)]$. For $n=3$ the present authors have proved [Bull. Amer. Math. Soc. Abstract 57-3-231] and for $n\geqq 3$ now conjecture that x_k-x^* is asymptotically a linear combination of the eigenvectors belonging to the largest and least eigenvalues of A^TA ($A^T=$transpose of A). (If there are eigenvectors orthogonal to x_0, disregard the corresponding eigenvalues.) When this asymptotic relationship holds for a given x_0, the sequence $\{(x_k-x^*)/|x_k-x^*|\}$ is asymptotically of period 2; then the usually slow convergence of x_k to x^* can be accelerated by occasionally determining the $x_{k+1}=\beta x_k+(1-\beta)x_{k-2}$ which minimizes $F[x_{k+1}(\beta)]$. Numerical experiments (on desk machines and I.B.M. card-programmed calculator) with a diagonal matrix $(n=36)$ and a general matrix $(n=6)$ confirm the conjecture and show the efficiency of the acceleration procedure. Compared to some other accelerations of iterative processes, this method has the advantages for automatic digital computation of being easily programmed and seeming to converge reasonably well. The accelerating shortcut can also be applied to the minimization of any well-behaved real-valued function F of the vector x; cf. the above reference. (Received March 12, 1951.)

Statistics and Probability

393. Z. W. Birnbaum and F. H. Tingey: *One-sided confidence contours for probability distribution functions.*

Let X be a random variable with a continuous probability distribution function $F(x) = \text{Prob.}\ \{X<x\}$. An ordered sample $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ of X defines the "empirical distribution function": $F_n(x)=0$ for $x<X_1$, $F_n(x)=k/n$ for $X_k \leqq x < X_{k+1}$, $k=1, 2, \cdots, n-1$, $F_n(x)=1$ for $X_n \leqq x$; the function $F_{n,\epsilon}(x) = \min\ \{F_n(x)+\epsilon, 1\}$ will be called "upper confidence contour." It is known that the probability $P_n(\epsilon) = \text{Prob.}\ \{F(x) \leqq F_{n,\epsilon}(x) \text{ for all } x\}$ is independent of $F(x)$. An expression for $P_n(\epsilon)$ was given by A. Wald and J. Wolfowitz in determinant form (Ann. Math. Statist. vol. 10 (1939) pp. 105–118), and an asymptotic expression for $n \to \infty$ is due to W. Feller (Ann. Math. Statist. vol. 19 (1948) pp. 177–189). In the present paper it is shown that $P_n(\epsilon) = 1 - \epsilon \sum_{j=0}^{[n(1-\epsilon)]} C_{n,j}(1-\epsilon-j/n)^{n-j}(\epsilon+j/n)^{j-1}$ for $0 \leqq \epsilon \leqq 1$. Furthermore, numerical solutions of the equation $P_n(\epsilon) = 1-\alpha$ are tabulated for $n=5, 8, 10, 20, 40, 50$ and $\alpha = .10, .05, .01, .001$, and it is observed that for $n=50$ these solutions agree very closely with those obtained from Feller's asymptotic expression for $P_n(\epsilon)$. (Received March 15, 1951.)

394. F. G. Tricomi: *On the statistical distribution of bacteria mutants.*

The study of the statistical distribution of bacteria which, thanks to a suitable *mutation*, become resistant to some antibiotica or other killing agents, leads to a new problem of calculus of probability, which can be considered as a far-reaching generalization of that leading to the well known Poisson distribution. In fact, under a certain hypothesis, the obtained distribution differs formally only slightly from that of Poisson. This hypothesis, however, is *not* satisfied in the bacteriological case which requires a very accurate treatment. In this case the author found the following (approximate) parametric representation of the frequency curve: $x=G(t)$, $y = \alpha^{1/2}[2\pi G'(t)]^{-1/2} \exp\ \{\alpha[G(t)-tG^*(t)]\}$. Here $\alpha>0$ is a parameter of the problem, $G(t)$ a new entire function defined by the expansion $G(t) = \sum_{n=1}^{\infty} t^n/[(2^n-1)n!]$, $G'(t)$ its derivative and $G^*(t)$ its integral over $(0, t)$. If α is small with respect to 1 (which is generally the case) the frequency curve has only a maximum at about $A \log (2A\alpha)$ where $A = \log_2 e = 1.44270$ and descends exceedingly quickly *on the left* of it. Furthermore the first moments of the distribution can be (rigorously) evaluated in any case, as well as a recurrence formula for the relevant probabilities. The new function $G(t)$ and some related ones are studied in a further paper of the author. (See Bull. Amer. Math. Soc. Abstract 57-4-387.) (Received March 12, 1951.)

Topology

395t. B. H. Arnold: *Topologies defined by bounded sets.*

Let S be a real vector space and $\mathcal{B}$ a subset of 2^S. The elements of $\mathcal{B}$ are called bounded subsets of S and are subjected to six intuitively satisfactory axioms. Example: The union of two bounded sets is bounded. These axioms completely determine the finite-dimensional bounded subsets of S as being the usual ones. Definition: A subset G of S is open if for each g in G, and B in $\mathfrak{B}$, there is a $\lambda>0$ such that $g+\lambda B\subset G$. This definition makes S a T_1 space; it may fail to be a Hausdorff space, and addition may fail to be continuous, but the cases where S is a normed linear space are easily characterized. If $\mathfrak{T}$ is the collection of all subsets of S which are topologically bounded, then $\mathfrak{T}\supset\mathfrak{B}$; the set $\mathfrak{T}$ is characterized in terms of $\mathfrak{B}$. The set $\mathfrak{T}$ satisfies the six basic axioms on bounded sets and the topologies in S generated by $\mathfrak{T}$ and $\mathfrak{B}$ are identical. Convergence in S is discussed and several examples are given, for example, a convergent directed system such that no cofinal subsystem is a bounded set. (Received March 12, 1951.)

396. E. G. Straus and F. A. Valentine: *A characterization of convex sets.*

Let S be a continuum in an n-dimensional Euclidean space R_n. *If S has the property that each point in S belongs to a unique maximal convex subset of S of dimension $\geqq n-1$, then S is convex.* This result, together with an obvious converse, yields a characterization of convex bodies in R_n. As an interesting corollary in R_2, we have: If S is a continuum in R_2 each point of which belongs to a unique maximal line segment of S, then S is a straight line segment (a segment contains at least two points). (Received March 15, 1951.)

397. R. B. Talmadge: *The representation of Baire functions.*

Let S be any topological space, $B(S)$ the set of real-valued Baire functions of class one on S. It is shown that *if S is normal, then $B(S)$ is isomorphic to the lattice of continuous functions on the Boolean space associated with the Boolean algebra of ambiguous sets of Borel class one.* Similar results are obtained for functions of class α in case S is a metric space. (Received March 9, 1951.)

398. P. A. White: *Extensions of the Jordan-Brouwer separation theorem and its converse.*

It is proved that if K is an $(n-1)$-generalized closed manifold ($n-1$-gcm) (see R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, 1949) (not necessarily connected) contained in the connected orientable n-gcm S, such that $S-K=A\cup B$ separate, then $\bar{A}$ and $\bar{B}$ are generalized manifolds with boundaries each consisting of some of the components of K. It is proved that if $A\subset S$ is open, uniformly-r-locally connected, $0\leqq r\leqq n-1$, and has an $(n-1)$-dimensional boundary, where S is a connected orientable n-gm, then $\bar{A}$ is an n-generalized manifold with boundary. This generalizes a similar result of Wilder where the additional hypothesis that the $(n-1)$-dimensional betti number of $S=0$ was included. To eliminate this hypothesis the following generalization of the Alexander duality theorem was proved: If K is a closed subset of the connected orientable n-gcm S such that $S-K$ has m components (m may be infinite), then $m-1\leqq p^{n-1}(K)\leqq(m-1)+p^{m-1}(S)$. (Received March 5, 1951.)

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