$$-\sum_{1}^{\infty} \frac{\lambda_n \cos nx + \mu_n \sin nx}{n^2}$$

and then "convolves" the series (22) with that of a testing function in the appropriate manner.

We have recounted all this with a view to suggesting that it would not be easy to decide what the general innovations in the present work are, analytical and even conceptual, and that it is in order to appraise the value of the book by its specific results, such as we have extracted above; and of such let the author produce many more, by all means.

S. BOCHNER

Tables relating to Mathieu functions. Characteristic values, coefficients, and joining factors. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. New York, Columbia University Press, 1951. 48+278 pp. $8.00.

Since its foundation (January 1938) the New York Unit of the Computation Laboratory of the National Applied Mathematics Laboratories, a division of the National Bureau of Standards—until July 1947 it was called the Mathematical Tables Project—has been very active in producing extensive and accurate numerical tables of important mathematical functions. Besides a series of tables of the elementary transcendents, they have published almost a dozen tables relating to the higher transcendents such as sine, cosine, and exponential integrals, probability functions, Bessel functions, Legendre functions. No matter how high-speed electronic calculating machinery may be further developed, applied mathematicians will always owe a great debt to these and other table-makers.

This particularly holds for the fascinating tables under review. As Professor Erdélyi emphasizes in the foreword, the comparatively slender numerical material available for Mathieu functions shows the urgency of the task undertaken by the National Bureau of Standards. The more so, since several important problems of applied mathematics and theoretical physics involving Mathieu functions have so far received only little attention because of lack of adequate numerical data. These problems include all types of vibrational, wave and diffusion problems connected with ellipses or elliptic cylinders, as well as stability investigations of various mechanical systems, the theory of frequency modulation, and loud-speaker theory.

Bessel functions and the like are rather elementary compared to Mathieu functions. The latter functions have no really simple representations such as power series with rational coefficients. On the other hand, a characteristic feature of Mathieu functions is that they are (in various ways) expansible into infinite series with a relatively simple three-term linear recurrence relation between successive coefficients. It is this feature (which the Mathieu functions share with the spheroidal wave functions) which renders possible the numerical calculation of Mathieu functions by the method of continued fractions. This technique was introduced by E. L. Ince and later on refined by others.

An excellent introduction, covering all the important numerical aspects of Mathieu functions and giving a wealth of material not easily to be found elsewhere, is presented by Gertrude Blanch of the Computation Laboratory. The standard form adopted for Mathieu's differential equation is (*) $y''+(b-s\cos^2 x)y=0$. Only real values of the parameters b and s are considered. It is known that (*) possesses periodic solutions of period π or 2π for the so-called characteristic values $b=\mathrm{be}_r$ or bo_r which are functions of s. These solutions are even or odd in x. There are altogether four different types of periodic Mathieu function, viz ($r=0, 1, 2, \cdots$)

$$\mathrm{Se}_{2r}(s, x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k}^{(2r)} \cos 2kx \qquad [\text{period } \pi,\ b = \mathrm{be}_{2r}(s)],$$

$$\mathrm{Se}_{2r+1}(s, x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k+1}^{(2r+1)} \cos (2k+1)x \qquad [\text{period } 2\pi,\ b = \mathrm{be}_{2r+1}(s)],$$

$$\mathrm{So}_{2r}(s, x) = \sum_{k=1}^{\infty} \mathrm{Do}_{2k}^{(2r)} \sin 2kx \qquad [\text{period } \pi,\ b = \mathrm{bo}_{2r}(s),\ r \neq 0],$$

$$\mathrm{So}_{2r+1}(s, x) = \sum_{k=0}^{\infty} \mathrm{Do}_{2k+1}^{(2r+1)} \sin (2k+1)x \qquad [\text{period } 2\pi,\ b = \mathrm{bo}_{2r+1}(s)].$$

The coefficients De and Do satisfy simple three-term recurrence relations. The characteristic values form an unbounded countably infinite set occurring in the order ($s>0$) $\mathrm{be}_0<\mathrm{bo}_1<\mathrm{be}_1<\mathrm{bo}_2<\mathrm{be}_2<\mathrm{bo}_3<\mathrm{be}_3<\cdots$. The periodic Mathieu functions are normalized by the conditions

$$(**) \qquad \mathrm{Se}(s, 0) = 1, \qquad \left[\frac{d}{dx}\mathrm{So}(s, x)\right]_{x=0} = 1.$$

The coefficients De and Do are sufficient to characterize the *complete* solution of (*) for $b=\mathrm{be}$ or bo. For details concerning the nonperiodic solutions corresponding to the characteristic values the reader should consult the introduction. The same holds for power series and asymptotic expressions for the characteristic values, relations between solutions corresponding to the parameters s and $-s$, differences in notation and normalization as compared to earlier work, etc.

There is one obvious disadvantage in adopting (*) as the standard Mathieu equation, so far as the characteristic values are concerned. It would have been more convenient to have a table of $a=b-s/2$, corresponding to McLachlan's canonical form $y''+(a-2q\cos 2x)y=0$, $s=4q$. The normalization (**) seems the most satisfactory, in particular with a view to extension of (**) to functions of unrestricted complex s.

The characteristic values $\mathrm{be}_r(s)$ are tabulated to eight decimals for the following values of r and s:

$r = 0$, $s = 0(0.2)20(0.5)37(1)100$; $r = 1$, $s = 0(0.2)15(0.5)50(1)100$;
$r = 2$, $s = 0(0.2)40(0.5)77(1)100$; $r = 3$, $s = 0(0.2)13(0.5)56(1)100$;
$r = 4(1)6$, $s = 0(0.5)100$; $r = 7(1)8$, $s = 0(1)100$; $r = 9(1)15$,
$s = 0(2)100$.

The characteristic values $\mathrm{bo}_r(s)$ are tabulated to eight decimals for

$r = 1(1)4$, $s = 0(0.5)100$; $r = 5(1)7$, $s = 0(1)100$; $r = 8(1)15$,
$s = 0(2)100$.

In both cases modified second differences are given. Two graphs of the characteristic values are included (stability chart). For negative s the characteristic values follow from $\mathrm{be}_{2r}(-s)=\mathrm{be}_{2r}(s)-s$; $\mathrm{bo}_{2r}(-s)=\mathrm{bo}_{2r}(s)-s$; $\mathrm{be}_{2r+1}(-s)=\mathrm{bo}_{2r+1}(s)-s$.

The greater part of the tables under review is devoted to the coefficients De_k and Do_k, at an interval in s sufficiently small to yield a set of key values of the functions Se_r and So_r amenable to subtabulation. (The Mathieu functions themselves are not tabulated.) The coefficients De_k of Se_r are tabulated for

$r = 0$, $s = 0(0.2)8(0.5)20(1)100$; $r = 1$, $s = 0(0.5)20(1)40(2)100$;
$r = 2$, $s = 0(0.2)8(0.5)15(1)40(2)100$; $r = 3$, $s = 0(0.2)20(1)40(2)100$;
$r = 4$, $s = 0(1)60(2)100$; $r = 5$, $s = 0(1)100$; $r = 6(1)7$, $s = 0(2)100$;
$r = 8$, $s = 0(4)68(2)100$; $r = 9$, $s = 0(4)100$; $r = 10$, $s = 0(5)100$;
$r = 11(1)15$, $s = 0(10)100$.

The coefficients Do_k of So_r are tabulated for

$r = 1$, $s = 0(0.5)10(1)40(2)100$; $r = 2$, $s = 0(1)40(2)100$;

$r = 3$, $s = 0(1)20(2)68(4)100$; $r = 4$, $s = 0(1)14(2)40(4)100$;

$r = 5$, $s = 0(2)60(4)100$; $r = 6$, $s = 0(2)100$; $r = 7(1)9$, $s = 0(4)100$;

$r = 10$, $s = 0(5)100$; $r = 11(1)15$, $s = 0(10)100$.

Most values are given to 9 or 10 decimals, though for the smaller values of k a constant number (9 or 10) of significant figures is provided so as to allow the complete solution of (*) to be calculated to a high degree of accuracy. The remaining tables contain joining factors and auxiliary functions, a description of which is beyond the scope of this review.

In conclusion, the reviewer thinks that the computation of the several Mathieu functions themselves would be very welcome. At least for the periodic Mathieu functions proper, this would not be too difficult for table-makers using high-speed electronic computers!

C. J. BOUWKAMP

Die zweidimensionale Laplace-Transformation. Eine Einführung in ihre Anwendung zur Lösung von Randwertproblemen nebst Tabellen von Korrespondenzen. By D. Voelker and G. Doetsch. (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, vol. 12.) Basel, Birkhäuser, 1950. 259 pp. 43 Swiss fr.

The motivation of this book is described by the authors in the preface in the following words. "The classical one-dimensional Laplace transformation now belongs to the common heritage of mathematicians and technologists, and since the publication of the monograph *Theorie und Anwendung der Laplace-Transformation*, books have been written on it in almost all cultured languages. By contrast, the two-dimensional (or double) Laplace transformation has been used only occasionally, in some memoirs. There is no systematic presentation of its theory and applications, or an elucidation of its distinctive features. Such a presentation is offered in this book which consists largely of unpublished material." (Reviewer's translation.)

The emphatic reference to the senior author's well known treatise should not mislead the reader into expecting the same sort of book here. On the one hand, the present book is based on Lebesgue's integration theory while Riemann integrals were used in Doetsch's book; and on the other hand, in contradistinction to the basic character of the earlier work, the orientation of the book under review is towards the applications, especially partial differential equations. This more practical orientation is shown by leaving aside basic ques-