Introduction to Hilbert space and the theory of spectral multiplicity. By P. R. Halmos. New York, Chelsea, 1951. 114 pp. $3.25.

There is little doubt that the author of this book enjoyed himself thoroughly during its preparation. Reading the result afforded this reviewer considerable pleasure. In one hundred and nine well-packed pages one finds an exposition which is always fresh, proofs which are sophisticated, and a choice of subject matter which is certainly timely. Some of the vineyard workers will say that P. R. Halmos has become addicted to the delights of writing expository tracts. Judging from recent results one can only wish him continued indulgence in this attractive vice.

The present work may confidently be recommended. However, beginners in the field should be cautioned before they rush off to secure a copy. Unless one is equipped and in training, one should not attempt the expedition. One must not be misled by the title. For this introduction to Hilbert space, one has to be an expert in measure theory. As a matter of fact it is best to have read the author's book on measure theory or its equivalent. One has to know enough about Banach spaces to be conversant with the Riesz representation theorem for the linear functionals on the space of continuous functions. We would be ready to wager that most young mathematicians learn that theorem subsequent to the theorem on the spectral resolution of hermitian operators and not prior to it—which is the scheme of things here. But, for those who know this material and wish an excellent introduction to multiplicity theory, this tract is just right.

The subject matter of the book is funnelled into three chapters: The geometry of Hilbert space; the structure of self-adjoint and normal operators; and multiplicity theory for a normal operator. For the last, an expert knowledge of measure theory is indispensable. Indeed, multiplicity theory is a magnificent measure-theoretic tour de force. The subject matter of the first two chapters might be said to constitute an introduction to Hilbert space, and for these, an a priori knowledge of classic measure theory is not essential. Thus one may question the author's decision to unveil his virtuosity in this direction sooner than was necessary, or perhaps desirable.

Chapter I has some features which differentiate it from previous texts in this domain. The Hilbert spaces under consideration are not assumed separable. The handling in proofs and notation from this point of view is completely successful. For another thing, considerable use is made of the parallelogram identity for vectors: $\|f+g\|^2+\|f-g\|^2=2\|f\|^2+2\|g\|^2$. The spirit of this and similar identities has certainly pervaded many smoke-filled colloquium rooms in the past decade and it is a pleasure to see it make its bow in book form. In terms of it certain proofs are very satisfactory. See, for example, the theorem concerning the existence of a vector perpendicular to a proper closed linear manifold.

As stated above, the second chapter is devoted to self-adjoint and normal operators. The bounded case only is treated. This calls for the usual preliminary development of a theory of projections. Also discussed are the elementary properties of the spectrum of an operator. In this connection the author gives a bizarre proof of the famous elementary theorem that if $\|1-A\|<1$, then A^{-1} exists. The series $(1-x)^{-1}=\sum_0^\infty x^n$ is probably the most extensively generalized in all of mathematics. Why not use it here instead of relying for a proof on Theorem 21.3, or any other for that matter? As mentioned before, the proof of the spectral theorem is based on "the external analytic crutch of measure theory." The author seems to regret (p. 71) "the lot of apparently formidable machinery" that he used; thus we shall not overburden his conscience with further allusion to the matter.

Before setting forth on the arduous journey through multiplicity theory, the reader is advised to be well-rested and of strong determination. Though the fundamental ideas and the ultimate result are quite reasonable and lucid, one's patience is often taxed by the annoying complexity inherent to the problem. Thus, such simple phenomena as those concerning orthogonality of manifolds can be compounded with such tirelessness as to wilt one's spirit. Before one reaches the point of no return in the proof, the temptation is great to let intuition become master over logic and consign mere proof to the antipurgatory. The original results in multiplicity theory were given by Hellinger in 1909. The treatment of non-separable spaces seems to be essentially more difficult. The first results and subsequent refinements are due to Wecken (1939), Nakano (1941), and Plessner and Rohlin (1946). It is these authors who have most influenced the present treatment. The problem here considered may be phrased in the form: Determine the unitary invariants of a single normal operator. That is, given two normal operators A and B, find significant, satisfactory, necessary, and sufficient conditions that for some unitary operator U, $B=U^{-1}AU$. For the finite-dimensional case, the answer is obviously that A and B should have the same spectral (or characteristic) values and that the manifold of vectors associated with each spectral value should have the same dimensionality for A as for B. The infinite-dimensional situation is enormously more complicated. The reader who is interested in a treatment which contains the present one as a special case, indeed, gives the unitary invariants of any commutative W^* algebra, should consult the recently published paper of I. E. Segal (Memoirs of the American Mathematical Society, no. 9, II, 1951).

The general result of the third chapter should be described briefly. Suppose that X is a space, S is a σ-Boolean algebra of sets over X, and that μ is a finite measure on S. Consider the Hilbert space $L_2(\mu)$ of square integrable functions on X. In this space one may construct for every set M in S the projection operator $E(M)f = \chi_M \cdot f$ where f is arbitrary in $L_2(\mu)$ and χ_M is the characteristic function of M. $E(M)$ is an example of a spectral measure. Now suppose u is a cardinal number. Form the direct sum of u copies of the space $L_2(\mu)$—this leads to phenomena of multiplicity u. Consider then the spectral measure defined by $E(M)\{f_k\} = \{\chi_M \cdot f_k\}$. Here $\{f_k\}$ is the general element in the direct sum space and k is an index ranging over a set of cardinal number u. Finally, let μ_j be orthogonal finite measures on S, for every j let u_j be a cardinal number, and construct the spectral measure $E(M)\{f_{jk}\} = \{\chi_M \cdot f_{jk}\}$. The latter is called the canonical example of a spectral measure.

Now let A be a normal operator. Then there is associated with A a complex resolution of the identity $E(\lambda)$ such that $A = \int \lambda dE(\lambda)$. If M is a Borel set in the complex plane, $E(M)$ is a spectral measure. The fundamental result of multiplicity theory is essentially that by a suitable isomorphism (unitary transformation between Hilbert spaces) this spectral measure is equivalent to a suitable canonical spectral measure. A critical and classical step in going from the spectral measure $E(M)$ associated with a normal operator A to an ordinary measure μ is the following: If x is a vector in the underlying Hilbert space $\mathfrak{H}$, then $\mu(M) = (E(M)x, x)$ is a finite measure. Here M is an arbitrary Borel set. Furthermore, if we write $U\chi_M = E(M)x$, then U may be extended to an isomorphism from the space $L_2(\mu)$ onto the closed linear manifold of $\mathfrak{H}$ generated by all elements of the form $E(M)x$. For this isomorphism U, one has $U^{-1}E(M)Uf = \chi_M \cdot f$. This fact is the building block for multiplicity theory.

The author's treatment of multiplicity theory seems to be quite satisfactory. Although the material is complicated, it is subdivided into pleasant compact packages which the reader absorbs as he proceeds on his journey. The author meanwhile is a veritable ringmaster who marshals his troupe of techniques, cracking the whip over each aspect of the theory in turn, thus keeping it at the proper pitch for its part. Several well written sections give the beginner a much desired heuristic approach to the situation.

The book closes with an informal discussion of source material and references and a bibliography containing 52 entries. Approximately one-third of these are vital to the present undertaking, the others being either marginal in value or representing a caprice of the author. No errors major or minor were detected, a fact which is only one indication of the very careful way in which the booklet was prepared. Most pages exhibit a zest for play as well as work which is refreshing. Indeed, at times one may have a vague apprehension that the author is preparing a prank or baiting a trap; however it seldom turns out to be more than a friendly tweak given with a wink. Such an intimate style, in the present desert of works written with an unexceptionable scientific detachment, is warmly welcome. It is certainly a facet to the general success enjoyed by Halmos' previous books.

E. R. LORCH

A theory of formal deducibility. By H. B. Curry. (Notre Dame Mathematical Lectures, no. 6.) University of Notre Dame, 1950. 9+126 pp.

The monograph contains a detailed account of the predicate calculus as presented by Gentzen (Math. Zeit. vol. 39 (1934) pp. 176–210, 405–431) in a sequence calculus in which the rules of inference follow in a natural way from the intended meanings of the logical connectives. However, Curry's treatment differs in several major respects from earlier ones. The predicate calculus is approached as an episystem over a basic formal system of elementary propositions. Various portions of the classical and intuitionistic systems are studied separately. There is a discussion of alternative concepts of negation. And a final chapter suggests a new approach to modal logic.

A formal system is specified by a primitive frame which defines inductively terms, elementary propositions, and theorems. (The author develops here notions presented in a paper in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 221–241.) In Curry's usage, the Hilbertian formal systems have as elementary propositions, propositions of the form "A is a provable formula," the formulas being terms in Curry's terminology. In studying a formal system it is customary and convenient not to limit attention only to elementary propositions, but to consider in addition compound propositions such as "Not for all formulas A, is A provable." These compound propositions are formed from the elementary ones by use of the logical connectives. Curry speaks of this broader system as an episystem over a formal system. (An episystem is not to be confused with a metasystem over