

THE OCTOBER MEETING IN NEW YORK

The four hundred ninety-fifth meeting of the American Mathematical Society was held on Thursday through Saturday, October 22–24, 1953, at Columbia University in New York City, in conjunction with a Conference on Training in Applied Mathematics sponsored jointly by the National Research Council and the Society. About 350 persons attended including the following 309 members of Society:

J. C. Abbott, Milton Abramowitz, C. R. Adams, M. I. Aissen, A. A. Albert, Linda Allegri, B. A. Amirà, R. D. Anderson, R. L. Anderson, M. M. Andrew, Joseph Andrushkiw, R. G. Archibald, P. N. Armstrong, N. J. Artin, W. A. Asprey, Silvio Aurora, W. G. Bade, Frederick Bagemihl, F. E. Baker, K. Y. Bal, R. P. Bambah, R. B. Barrar, J. H. Barrett, L. K. Barrett, Walter Bartky, E. G. Begle, A. A. Bennett, R. R. Bernard, Lipman Bers, M. A. Biot, I. E. Block, H. W. Bode, H. F. Bohnenblust, S. G. Bourne, J. W. Bower, J. R. Bowman, E. H. Boyle, J. W. Brace, C. C. Bramble, Eleazer Bromberg, F. E. Browder, A. A. Brown, A. B. Brown, J. L. Brown, Jr., F. H. Brownell, R. S. Burington, C. W. Burrill, S. S. Cairns, Lorenzo Calabi, F. P. Callahan, Jr., E. W. Cannon, E. D. Carey, Abraham Charnes, K. L. Chung, S. I. Ciolkowski, Paul Civin, F. E. Clark, E. J. Cogan, L. W. Cohen, R. M. Cohn, T. F. Cope, Richard Courant, J. H. Curtiss, M. D. Darkow, M. D. Davis, Philip Davis, R. B. Davis, Allen Devinatz, R. J. DeVogelaere, A. H. Diamond, D. J. Dickinson, V. J. Doberly, H. L. Dorwart, Avron Douglis, L. A. Dragonette, D. M. Dribin, Nelson Dunford, Jacques Dutka, Aryeh Dvoretzky, H. A. Dye, B. E. Dyer, H. P. Edmundson, Bernard Epstein, D. I. Epstein, Paul Erdős, G. C. Evans, H. P. Evans, W. H. Fagerstrom, Michael Fekete, J. M. Feld, W. E. Ferguson, N. J. Fine, C. D. Firestone, Irwin Fischer, D. A. Flanders, William Forman, R. M. Foster, Phyllis Fox, R. H. Fox, Gerald Freilich, Bernard Friedman, H. D. Friedman, Orrin Frink, T. C. Fry, H. K. Fulmer, I. N. Gál, I. S. Gál, J. J. Gergen, Murray Gerstenhaber, H. A. Giddings, Wallace Givens, Sidney Glusman, J. K. Goldhaber, Saul Gorn, Arthur Grad, Harold Grad, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, V. B. Haas, P. R. Halmos, M. E. Hamstrom, G. H. Handelman, G. A. Hedlund, I. N. Herstein, T. H. Hildebrandt, Abraham Hillman, W. M. Hirsch, S. B. Hobbs, P. G. Hodge, Jr., S. P. Hoffman, Banesh Hoffmann, F. E. Hohn, S. S. Holland, Jr., H. G. Hopkins, Gabriel Horvay, D. B. Houghton, B. K. Hovey, C. C. Hsiung, A. O. Huber, Witold Hurewicz, Eugene Isaacson, J. R. Isbell, W. S. Jar-detzky, R. V. Kadison, Shizuo Kakutani, Aida Kalish, E. L. Kaplan, Irving Kaplansky, D. E. Kibbey, S. G. Kneale, George Klein, Morris Kline, Daniel Kocan, I. I. Kolodner, B. O. Koopman, A. G. Kostenbauder, Saul Kravetz, J. B. Kruskal, Jr., M. D. Kruskal, Harold W. Kuhn, O. E. Lancaster, R. E. Langer, C. W. Langley, P. D. Lax, Solomon Leader, E. H. Lee, Solomon Lefshetz, R. A. Leibler, W. W. Leutert, Howard Levene, M. E. Levenson, H. M. Lieberstein, S. C. Lowell, G. S. S. Ludford, Eugene Lukacs, Harold Luxenberg, B. H. McCandless, S. W. McCuskey, Brockway McMillan, E. J. McShane, L. A. MacColl, D. B. MacMillan, H. M. Mac-Neille, Wilhelm Magnus, A. J. Maria, M. H. Maria, Lawrence Markus, M. H. Martin, F. J. Massey, C. W. Marshall, F. H. Miller, K. S. Miller, H. J. Miser, Deane Montgomery, Morris Morduchow, B. J. Morse, David Moskovitz, C. A.

Nelson, D. J. Newman, Louis Nirenberg, A. B. Novikoff, C. O. Oakley, G. G. O'Brien, Alex Orden, J. C. Oxtoby, J. S. Oxtoby, L. J. Paige, Emanuel Parzen, L. E. Payne, E. J. Pellicciaro, Anna Pell-Wheeler, A. S. Peters, W. A. Pierce, H. O. Pollak, William Prager, Hans Rademacher, M. S. Rees, Haim Reingold, Russell Remage, Helene Reschovsky, Moses Richardson, P. D. Ritger, E. K. Ritter, I. F. Ritter, Herbert Robbins, Mark Robinson, Robin Robinson, Selby Robinson, I. H. Rose, M. E. Rose, P. C. Rosenbloom, J. E. Rosenthal, M. F. Roszkopf, J. P. Russell, E. A. Saibel, H. E. Salzer, J. E. Sammet, W. C. Sangren, Arthur Sard, Herbert Scarf, H. M. Schaerf, A. T. Schafer, R. D. Schafer, Miriam Schapiro, S. A. Schelkunoff, E. V. Schenkman, M. M. Schiffer, Albert Schild, K. C. Schraut, Seymour Schuster, Abraham Schwartz, C. H. W. Sedgewick, I. E. Segal, Wladimir Seidel, H. N. Shapiro, V. L. Shapiro, I. M. Sheffer, I. M. Singer, James Singer, M. L. Slater, David Slepian, O. K. Smith, J. J. Sopka, J. J. Stoker, Walter Strodt, Sister M. Carmel Sullivan, M. M. Sullivan, R. L. Swain, Alexander Tartler, J. T. Tate, A. H. Taub, Olga Taussky, T. Y. Thomas, D. L. Thomson, Jr., John Todd, L. F. Tolle, M. L. Tomber, C. B. Tompkins, A. W. Tucker, J. W. Tukey, H. E. Vansant, H. V. Waldinger, Jack Warga, J. H. Weiner, Alexander Weinstein, Louis Weisner, Bernard Weitzer, David Wellinger, John Wermer, F. J. Weyl, A. L. Whiteman, Hassler Whitney, Albert Wilansky, A. B. Willcox, Sister Teresa Dolores Wolf, Jacob Wolfowitz, K. G. Wolfson, E. S. Wolk, V. M. Wolontis, Arthur Wouk, F. B. Wright, Jr., Michael Yanowitch, D. M. Young, L. A. Zadeh, Arthur Zeichner, Leo Zippin, O. J. Zobel.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Paul Erdős of the University of Notre Dame addressed a general session Saturday afternoon on *Combinatorial problems in set theory*. Professor Kakutani presided.

Sessions for contributed papers were held Saturday morning, Professor Kaplansky presiding, and afternoon, Professors Oxtoby and Schiffer presiding.

The program of the conference on Training in Applied Mathematics consisted of papers invited by the National Research Council Committee on Applied Mathematics and distributed in sessions Thursday afternoon, Friday morning and afternoon, Saturday morning as follows:

Sessions on Applied Mathematics in the Traditional Departmental Structure: Professors G. C. Evans, University of California; T. H. Hildebrandt, University of Michigan; R. E. Langer, University of Wisconsin.

Session on The Role of Mathematics in the Integrated School of Applied Science: Professors J. B. Wiesner, Massachusetts Institute of Technology; H. W. Emmons, Harvard University; A. H. Taub, University of Illinois.

Session on The Graduate Institute for Applied Mathematics: Professor Richard Courant, Institute for Mathematics and Mechanics, New York University; Professor William Prager, Brown Univer-

sity; Professor M. M. Schiffer, Stanford University; Professor M. H. Martin, University of Maryland.

Session on Applied Mathematics Training in Europe: Dr. F. J. Weyl, Office of Naval Research, Chairman; Professor G. C. McVittie, University of Illinois; Professor P. Germain, Université de Poitiers; Professor A. van Wijngaarden, Mathematisch Centrum, Amsterdam.

Session on The Mathematician in Government Establishments: Dean Mina S. Rees, Hunter College; Dr. C. C. Bramble, Naval Proving Ground, Dahlgren; Dr. W. W. Leutert, Aberdeen Proving Ground; Dr. H. J. Miser, USAF Operations Analysis Division; Dr. C. B. Tompkins, Institute for Numerical Analysis, Los Angeles.

Session on The Mathematician in Industrial Organizations: Dean Walter Bartky, University of Chicago; Dr. E. C. Nelson, Hughes Aircraft Company; Dr. A. A. Brown, A. D. Little, Inc.; Dr. H. Gerschinowitz, Shell Development Company; Dr. H. W. Bode, Bell Telephone Laboratories.

General Session: Summary Address, Dr. T. C. Fry, Bell Telephone Laboratories; Professor Leon W. Cohen, National Science Foundation and Queens College, presiding.

The abstracts of contributed papers presented at the meeting follow. Those with "t" after the abstract number were presented by title. Where a paper with joint authorship was presented in person, (p) follows the name of the author presenting it. Dr. Schneider was introduced by Dr. I.T.A.C. Adamson.

ALGEBRA AND THEORY OF NUMBERS

1t. Leonard Carlitz: *A problem involving quadratic forms in a finite field.*

Let $g = p^n$, $p > 2$, and let A denote a nonsingular symmetric matrix of order m with elements in $GF(q)$. We determine the number of $m \times t$ matrices such that $|\begin{smallmatrix} A & X \\ X^t & 0 \end{smallmatrix}| = \beta$, where $\beta \in GF(q)$ is fixed. Application is made to the evaluation of a generalized Gauss sum. (Received August 24, 1953.)

2t. Leonard Carlitz: *Certain special equations in a finite field.*

The equation $a_1x_1^2 + \dots + a_r x_r^2 = bx_1x_2 \dots x_r + c$, where $a_i, x_i, b, c \in GF(q)$, is discussed for $r=3, 4$. In the case $r=3$ a simple explicit formula is found for the number of solutions. In the case $r=4$ the number of solutions is expressed in terms of a Jacobsthal sum; in certain special cases explicit results are obtained. (Received August 24, 1953.)

3t. Leonard Carlitz: *Representations by skew forms in a finite field.*

Let $q = p^n$, $p > 2$, and let A and B denote skew-symmetric matrices of order m and t , respectively, with elements in $GF(q)$. Explicit formulas are found for the number of $m \times t$ matrices X such that $X'AX = B$. As an application the number of solutions $X = X(m, t)$ of the equation $\begin{vmatrix} x & x \\ x' & 0 \end{vmatrix} = \beta$ is determined, where A is skew-symmetric and nonsingular. (Received August 24, 1953.)

4t. Leonard Carlitz: *Sets of primitive roots.*

The following result is typical. Let $f_1(x), \dots, f_r(x)$ denote quadratfrei polynomials with coefficients in $GF(q)$ that are relatively prime in pairs and of degree ≥ 1 ; let N_r denote the number of α 's in $GF(q)$ such that $f_i(\alpha)$ is a primitive root of $GF(q)$ for $i = 1, \dots, r$. Then $N_r \sim \phi^r(q-1)/q^{r-1}$, $q \rightarrow \infty$. (Received August 24, 1953.)

5. V. J. Doberly (Dobroliuboff): *Doberly's logarithmic series.*

For preparation of mathematical tables and solution of problems involving direct calculation of natural, denary, and other logarithms, Doberly's logarithmic series provide a larger scope of usefulness and practicability, because, unlike any other known logarithmic series, they permit the use of unlimited number of different infinite progressions—all of different convergibility—through which to calculate the logarithmic number. Of many series of the kind those applicable to calculation of the natural logarithms are the simplest. Of them the series $\log_{\text{nat}} x = m + (x/e^m - 1) - (1/2)(x/e^m - 1)^2 + (1/3)(x/e^m - 1)^3 - (1/4)(x/e^m - 1)^4 + \dots$, where m is any number satisfying the condition $m > \log_{\text{nat}} x/2$, and $\log_{\text{nat}} x = m + ((x - e^m)/x) + (1/2)((x - e^m)/x)^2 + (1/3)((x - e^m)/x)^3 + (1/4)((x - e^m)/x)^4 + \dots$, where $m < \log_{\text{nat}} 2x$, are especially valuable. When the logarithm sought is approximately known, the substitution of the approximate value of $\log x$ for m in the Doberly's series makes them strongly convergent, thus virtually permitting one to obtain a sufficiently precise figure of the logarithm by discarding in the computation all but the few first terms of the series. (Received August 4, 1953.)

6. I. N. Herstein: *On the Lie and Jordan rings of a simple associative ring.*

Given any associative ring A , to it one can attach two other rings, possibly no longer associative, A^J the Jordan ring, and A^L the Lie ring of A by defining new products for the elements of A , $a \cdot b = ab + ba$ to yield A^J , and $[a, b] = ab - ba$ to yield A^L . The following results are proved: (1) If A is a simple ring of characteristic $\neq 2$, then A^J is a simple Jordan ring. (2) If A is a simple, non-nil ring of characteristic $\neq 2$, then any Lie ideal U of A^L is either contained in the center of A or must contain $[A, A] = \{ab - ba \mid a, b \in A\}$. (Received September 11, 1953.)

7. J. R. Isbell: *More on the continuity of the real roots of an algebraic equation.*

Hewitt announced in *Rings of real-valued continuous functions*. I. Trans. Amer. Math. Soc. vol. 64, the following theorem: Let $C(X, R)$ be the ring of all continuous functions on completely regular X to the real line R ; let M be a maximal ideal in $C(X, R)$. The residue field $C(X, R)/M$ is real closed. However, Hewitt's proof that a polynomial of odd degree has a root is defective. Henriksen and Isbell, *On the continuity of the real roots of an algebraic equation*, Proc. Amer. Math. Soc. vol. 4, restored the proof with use of the Tietze extension theorem and thus only in

case X is normal. A new proof is given here which avoids extension and thus establishes the theorem as Hewitt stated it. (Received September 21, 1953.)

8t. Bjarni Jónsson: *Representations of lattices. I.*

Whitman showed that every lattice L is isomorphic to a lattice L' of equivalence relations. If we regard binary relations as sets of ordered pairs, then multiplication and inclusion in the lattice L' coincide with set-theoretic intersection and inclusion while the lattice sum of two relations R and S in L' is the set-theoretic union of the nondecreasing infinite sequence of relative products $R; S, R; S; R; S; R; S; R; S; \dots$. If for every R and S in L' this sequence is constant from the n th term on, then we speak of a representation of L of type n . *Results:* Every lattice L has a representation of type 3. In order for L to have a representation of type 2, it is necessary and sufficient that L be modular. Previously we have shown that there exist modular lattices which do not have a representation of type 1 (a representation by commuting equivalence relations), cf. Bull. Amer. Math. Soc. Abstract 59-4-341. (Received September 8, 1953.)

9t. Bjarni Jónsson: *Representations of lattices. II. Preliminary report.*

Consider the four classes (i) \mathcal{E} , (ii) \mathcal{N} , (iii) \mathcal{A} , (iv) \mathcal{L} , consisting, respectively, of all modular lattices which can be represented isomorphically by means of (i) commuting equivalence relations, (ii) normal subgroups of a group, (iii) subgroups of an Abelian group, (iv) subspaces of a (possibly degenerate) Desarguesian projective space. Let \mathcal{C} be the class of all complemented modular lattices and \mathcal{D}_n the class of all n -dimensional modular lattices. *Results:* 1. $\mathcal{E} \cap \mathcal{C} = \mathcal{N} \cap \mathcal{C} = \mathcal{A} \cap \mathcal{C} = \mathcal{L} \cap \mathcal{C}$. 2. $\mathcal{E} \cap \mathcal{D}_n = \mathcal{N} \cap \mathcal{D}_n = \mathcal{A} \cap \mathcal{D}_n = \mathcal{L} \cap \mathcal{D}_n$ for $n \leq 4$. 3. $\mathcal{E} \cap \mathcal{D}_6 \neq \mathcal{N} \cap \mathcal{D}_6$. 4. If $L \in \mathcal{E}$, then L satisfies the following condition (α): For any $a_0, a_1, a_2, b_0, b_1, b_2 \in L$, if $y = (a_0 + a_1) \cdot (b_0 + b_1) [(a_0 + a_2)(b_0 + b_2) + (a_1 + a_2)(b_1 + b_2)]$, then $(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0(a_1 + y) + b_0(b_1 + y)$. 5. If $L \in \mathcal{C}$ or $L \in \mathcal{D}_n$ with $n \leq 4$, and if the condition (α) holds, then $L \in \mathcal{E}$. 6. Suppose $L \in \mathcal{C}$ is of order $k \geq 3$ (in the sense of von Neumann). Then L is isomorphic to the lattice of all principal right ideals of a regular ring if and only if (α) holds. (Received September 8, 1953.)

10t. Jakob Levitzki: *Contributions to the theory of nilrings.*

We introduce four new types of nilrings which are encountered in the theory of iterated representations (C. J. Everett, Jr., Duke Math. J. vol. 5 (1939)) in case the iteration is transfinite, and also in the theory of transfinitely iterated socles (R. Baer, *Radical ideals*, Amer. J. Math. vol. 65 (1943)). The method by which these rings are defined is extended to enable a useful classification of the various hitherto known types of nilrings. A ring S is right (left) vanishing, in short, a v_r -ring (v_l -ring), if for any infinite sequence (a_i) in S there are an n and an m such that $a_n a_{n+1} \cdots a_{n+m} = 0$ (resp. $a_{n+m} a_{n+m-1} \cdots a_n = 0$), and it is a v_r -ring if it is both a v_r -ring and a v_l -ring. If above condition holds only for sequences of type ${}^* \omega + \omega$, the ring is called a v^* -ring. One recalls that a ring S is called an L -ring if it coincides with its lower radical. One has the following implications: Nilpotent rings $\rightarrow v_r$ -rings $\rightarrow v_l$ -ring (v_l -rings) $\rightarrow v^*$ -rings $\rightarrow L$ -rings, and all these types of nilrings are distinct. A v_l -ring is consumed by a transfinite ascending chain of iterated right annihilators culminating in the ultimate r -annihilator $M_r^{(r)}$. The ring $S - M_r^{(r)}$ is l . definite (i.e. its left representation is an isomorphism) and $M_r^{(r)} = \text{intersection of all } A \text{ such that } S - A \text{ is } l. \text{ definite. } S = M_r^{(r)}$ if

and only if S is a v_l -ring, S is transfinitely right nilpotent and $\bar{s}^{r+1}=0$, while for a v^* -ring the ultimate kernel is equal to 0. (Compare J. Levitzki, *Powers with transfinite exponents*. II, Riveon Lematematika vol. 2 (1947)). (Received September 14, 1953.)

11t. F. R. Olson: *Some determinants involving Bernoulli and Euler numbers of higher order.*

Determinants whose elements are Bernoulli, Euler, and related numbers of higher order are evaluated. Using the notation of Nörlund, *Differenzrechnung*, Chap. 6, it is shown, for example, that $|B_i^{(a+jd)}| = \prod_{k=0}^m (-d/2)^k k!$ for a, d constants; $i, j = 0, 1, \dots, m$. Like results are obtained for the other numbers. (Received August 24, 1953.)

12. R. D. Schafer (p) and M. L. Tomber: *A new simple Lie algebra of characteristic 2.*

Let \mathbb{C} be any Cayley algebra over a field Φ of characteristic 2, and \mathfrak{F} be the 27-dimensional space of all those 3×3 hermitian matrices X with elements in \mathbb{C} which have scalar elements in the principal diagonal. Then \mathfrak{F} is proved to be a (restricted) Lie algebra of characteristic 2 relative to the composition $X \circ Y = XY + YX$. The derived algebra \mathfrak{F}_0 (the set of all X of trace 0) is a new simple (restricted) Lie algebra, being central simple and of dimension 26 over Φ . It is shown that the derivation algebra \mathfrak{D} of \mathfrak{F} is 52-dimensional, the outer derivation algebra $\mathfrak{D}/\text{ad } \mathfrak{F}_0$ being simple. If \mathbb{C} is assumed to have divisors of zero, as would be the case if Φ were assumed algebraically closed, it is shown that $\mathfrak{D}/\text{ad } \mathfrak{F}_0 \cong \mathfrak{F}_0$. (Received July 30, 1953.)

13t. Hans Schneider: *The null ideal belonging to a commutative ring of linear transformations.*

Let V be a finite-dimensional vector space over a commutative field k . Let a_1, \dots, a_m be commutative linear transformations on V . Let $R = k[x_1, \dots, x_m]$, where x_1, \dots, x_m are indeterminates, and let $A = k[a_1, \dots, a_m]$. The null ideal \mathfrak{n} belonging to A is defined to be the kernel of the natural homomorphism ϕ of R onto A . Let $\mathfrak{n} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ be the short representation of \mathfrak{n} as the intersection of primary ideals. It is proved that the prime ideals belonging to $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are maximal. Let Q_i be the image of \mathfrak{q}_i under ϕ . It is pointed out that V is also an A -space and that $V = V_1 \oplus \dots \oplus V_s$, where V_i is the A -subspace of V annihilated by Q_i . The ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are prime if and only if V is completely A -reducible. The space V is the direct sum of A -subspaces, which are one-dimensional over k , if and only if A is isomorphic to a ring of diagonal matrices with coefficients in k . Thus if A is isomorphic to such a ring, then V is completely A -reducible. The reverse implication holds when k is algebraically closed. (Received July 30, 1953.)

14t. Hans Zassenhaus: *Invariant bilinear forms on Lie algebras.*

Any representation Δ of a Lie algebra L over a field F determines uniquely the symmetric bilinear form $(a, b) = \text{tr } (\Delta a \cdot \Delta b)$ on L satisfying the invariance condition $(a \circ b, c) = (a, b \circ c)$. For an ideal A of L the set κA of all solutions x of $(x, A) = 0$ is an ideal of L . Also the set $L^{-1} \circ A$ of all solutions x of $L \circ x \subseteq A$ is an ideal of L . Theorem 1: $\kappa(L^{-1} \circ (A + \kappa L)) = L \circ A + \kappa L$, $\kappa(L \circ A) = L^{-1} \circ (A + \kappa L)$. Theorem 2: For a nilpotent subalgebra H coinciding with its normalizer in L , $H \circ H \subseteq \kappa L$. Theorem

3: Let F be algebraically closed, $\kappa L = 0$, and let $D^2L = DL$ if $\text{char. } F = 2$. Then L is the algebraic sum of r mutually orthogonal Lie algebras L_i which are either simple or, only if $\text{char. } F \neq 0$, $L_i \circ L_i/z(L_i)$ is simple nonabelian and $\dim z(L_i) = \dim L_i/L_i \circ L_i = 1$. If Δ is irreducible then $r = 1$. The proofs make use of the representation theory of nilpotent Lie algebras developed by the author earlier. Application is made to simplicity proofs for classical Lie algebras. Symmetric bilinear forms on an F -submodule of L invariant under a set of derivations of L are set in 1-1 correspondence with certain elements of degree 2 in the Birkhoff-Witt algebra of L , if the form is nondegenerate and $\text{char. } F \neq 2$. (Received August 24, 1953.)

ANALYSIS

15. Joseph Andrushkiw: *A note on polynomials all of whose zeros are real and of the same sign.*

Let all the zeros of the polynomial $f(z) = f(z; 1) = 1 + a_1z + \dots + a_nz^n$ be negative. Consider the polynomial $f(z; x) = 1 + a_1^x z + a_2^x z^2 + \dots + a_n^x z^n$, x real. Its discriminant expressed in terms of coefficients is a continuous function of x . From theorem of Schurmalo (e.g. J. Schur, *Zwei Sätze über alg. Gleichungen mit lauter reellen Wurzeln*, J. Reine Angew. Math. vol. 144 (1914)) and the properties of a polynomial with real zeros only (author's paper, *On the zeros of integral functions*, Bull. Amer. Math. Soc. Abstract 58-3-289) it follows that the above discriminant has a greatest positive zero X_0 . For $x \geq X_0$ all the zeros of $f(z; x)$ are negative. On the other hand, if $0 < x_0 = \max [\log ((r+1)/(r+2))/\log (a_{r+1}^2/a_r a_{r+2})]$, $r = 0, 1, \dots, n-2$, the polynomial $f(z; x)$ has at least one pair of imaginary zeros for $x \leq x_0$. (Received September 8, 1953.)

16t. W. G. Bade: *Bounded Boolean algebras of projections.*

Let X be a reflexive Banach space and \mathfrak{B} be a bounded Boolean algebra (B.A.) of projections in X (for some M , $\|E\| \leq M$, $E \in \mathfrak{B}$). \mathfrak{B} is complete if for every subset $\{E_\alpha\} \subseteq \mathfrak{B}$, $X = M \oplus N$ where $M = \text{cls } \{E_\alpha X\}$, $N = \bigcap_\alpha (I - E_\alpha)X$, and \mathfrak{B} contains the projection $E = \bigvee_\alpha E_\alpha$ defined by this decomposition. A complete bounded B.A. of projections in X contains every projection in its weak closure. If X is separable, a σ -complete bounded B.A. of projections is complete. The algebra generated by a complete bounded B.A. of projections is weakly closed. A new proof is given of a theorem of N. Dunford (Bol. Soc. Mat. Mexicana vol. 3 (1946)) that the closure of \mathfrak{B} in the strong operator topology is the least complete B.A. containing \mathfrak{B} . (Received September 9, 1953.)

17. W. G. Bade: *Weakly closed algebras of spectral operators and strong limits.*

Let \mathfrak{A} be a commutative algebra of scalar type spectral operators in a reflexive Banach space (see N. Dunford, Bull. Amer. Math. Soc. Abstract 59-2-185 for definitions). If the Boolean algebra \mathfrak{B} of projections generated by the resolutions of the identity for elements of \mathfrak{A} is bounded, then every element of $\overline{\mathfrak{A}}^w$ (weak closure) is a scalar type spectral operator. If \mathfrak{A} is generated (uniform operator topology) by \mathfrak{B} , then $\overline{\mathfrak{A}}^w$ is generated by $\overline{\mathfrak{B}}^s$ (strong closure). If X is separable, then $\overline{\mathfrak{A}}^w$ is generated by the resolution of the identity of a single element. This result generalizes a theorem of von Neumann for commutative W^* -algebras in Hilbert space. With a suitable restriction on the spectra of the limiting operators, a limit in the strong operator topology of scalar type spectral operators is of this type without commuta-

tivity. For spectral operators not of scalar type a strong limit need not be spectral. (Received September 9, 1953.)

18*t.* Frederick Bagemihl and Wladimir Seidel: *Spiral and other asymptotic paths, and paths of complete indetermination, of analytic and meromorphic functions.*

Let λ , μ , and ν be non-negative integers with $1 \leq \nu = \lambda + \mu$, and in the complex plane let G be a region not containing the point ∞ whose boundary consists of ν mutually exclusive circles K_1, K_2, \dots, K_ν , some or all of which may degenerate to single points. If $\lambda > 0$, let $\{S_n\}$ be an arbitrary enumerable set of mutually exclusive paths (which may be spirals) in G , each of which converges monotonically to one of the K_j , and $\{w_n\}$ be a sequence of complex numbers, some or all of which may be infinite. Then there exists a single-valued nonconstant function $f(z)$ in G , regular if $\lambda = \nu = 1$, or if $\mu = 0$ and no K_j is degenerate, but meromorphic otherwise, such that, if $\lambda > 0$, $f(z)$ possesses the asymptotic value w_n along S_n for every n , and, if $\mu > 0$, every path in G converging to any one of the circles $K_{\lambda+1}, \dots, K_{\lambda+\mu}$ is a path of complete indetermination of $f(z)$ (i.e., is mapped by $f(z)$ on a set which is everywhere dense in the complex plane). This sharpens or generalizes results of Gross [Monatshefte für Mathematik und Physik vol. 29 (1918) p. 14] and Valiron [C.R. Acad. Sci. Paris vol. 198 (1934) p. 2065]. If $f(z)$ is nonconstant and meromorphic in $|z| < 1$, and $\sum(1 - |z_n|) = \infty$ for the zeros z_n of $f(z)$ in every sector of $|z| < 1$, then $f(z)$ has a residual set of radii of complete indetermination. Other sufficient conditions are obtained for paths of complete indetermination. (Received September 11, 1953.)

19. Lipman Bers: *An application of function theory to quasi-linear elliptic equations.*

The following lemma is established with the aid of the uniformization theorem, Privaloff's theorem on conjugate functions, and the Ahlfors-Lavrentyeff theorem on quasi-conformal homeomorphisms: Let $w(z)$ be a quasi-conformal function defined in a domain D with sufficiently smooth boundary. Assume that the component of w in a given Hölder-continuously varying direction satisfies on the boundary a Hölder condition. Then w satisfies a Hölder condition on the closure of D . Since the gradient $w = \phi_x - i\phi_y$ of a function $\phi(x, y)$ satisfying an elliptic equation (1) $A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} = 0$ is a quasi-conformal function, the lemma leads to a priori estimates for the modulus and Hölder continuity of w under the assumption that ϕ satisfies appropriate boundary conditions. This method, which is independent of the so-called "three point condition," leads to a variety of existence theorems for nonlinear equations of the form (1), A, B, C being functions of $x, y, \phi, \phi_x, \phi_y$. In particular, multiply-connected and unbounded domains, and boundary conditions other than the first, can be treated. In the present paper only the main idea of the method is presented; detailed applications will be given in subsequent publications. (Received September 9, 1953.)

20*t.* Lipman Bers: *Local theory of pseudo-analytic functions.*

In this paper the local theory of pseudo-analytic functions (L. Bers, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 130-136, vol. 37 (1951) pp. 42-47 and *Theory of pseudoanalytic functions*, mimeographed notes, New York University, 1953) is extended to the case in which the generators may be nondifferentiable but are merely assumed to satisfy a Hölder condition. The theory now applies to any linear homo-

geneous elliptic system of two first-order equations for two unknown functions of two variables, with Hölder-continuous coefficients. The extension of the global theory to this case will be considered elsewhere. (Received September 9, 1953.)

21t. R. P. Boas: *Interference phenomena for entire functions.*

Let $f(z)$ be an entire function of exponential type π which is $o(|x|)$ on the real axis and which is bounded at the positive and negative integers. S. Bernstein, [Izvestiya Akad. Nauk SSSR. Ser. Mat. vol. 12 (1948) pp. 421–444] proved that the combination $f(x+1/2)+f(x-1/2)$ is bounded on the real axis, and A. F. Timan [Doklady Akad. Nauk SSSR. vol. 89 (1953) pp. 17–20] showed more generally that $o(|x|)$ in the hypothesis can be replaced by $o(|x|^q)$ and that the more general transform $\int_{-\infty}^{\infty} f(x+t)d\rho(t)$ is always bounded under the condition $\int_{-\infty}^{\infty} e^{\sigma|x|} |d\rho(x)| < \infty$ for some $\sigma > \pi$, if and only if $\int_{-\infty}^{\infty} e^{\pm i\pi t} d\rho(t) = 0$. Timan also gave a more general result when $f(x) = o(|x|^q)$, $q > 2$. In this paper it is shown that (i) if $f(x) = o(|x|)$ and $\{f(n)\}$ is bounded, and if $\lambda(z) = \sum \lambda_n z^n$ is regular for $|z| \leq \pi$, then $\lambda(D)f(z) = \sum \lambda_n f^{(n)}(z)$ is always bounded on the real axis if and only if $\lambda(i\pi) = \lambda(-i\pi) = 0$; (ii) if $\{f(n)\}$ is bounded and $f(x) = o(|x|^q)$, $q > 1$, then $f(z)$ is of the form $g(z) + P(z) \sin \pi z$, where $g(x) = o(|x|)$ and $P(z)$ is a polynomial. Timan's results follow from these facts. (Received August 10, 1953.)

22t. D. G. Bourgin: *Some mapping theorems.*

Denote by X a point symmetric set in R^n , the Euclidean n -space, which carries a symmetric m cycle. Require that this m cycle, on identification of antipodal points, be nonbounding in Q , the product of a projective n -space and the unit segment. Let f map X into R^j . As generalizations of theorems of Borsuk and of Dyson it is stated: (A) The symmetric set, each of whose points is assigned the same point of R^j as its antipode, carries an $n-j$ cycle which on identification of antipodal points does not bound in Q . (B) There are $m-j+1$ symmetric orthogonal segments whose vertices lie in X and map into some $j-1$ sphere about the origin in R^j . Continuity of the map may be weakened to upper semi-continuity of the functions $f_i(x) - f_i(-x)$, where $f_i(x)$ is the i th coordinate of $f(x)$. (Received September 4, 1954.)

23t. F. E. Browder: *The asymptotic distribution of the eigenvalues of the non-self-adjoint elliptic differential operator.*

If D is a bounded domain in E^n , let K be a suitably differentiable linear elliptic differential operator on D , B a suitably differentiable linear differential operator of a single sign on D . Generalizing a result due to Carleman for second-order operators in three independent variables, the asymptotic distribution of the eigenvalues of K with respect to B on D is determined in terms of the number of independent variables as well as the orders and coefficients of K and B . K is not assumed to be self-adjoint. (Received September 15, 1953.)

24. F. H. Brownell: *An extension of Weyl's asymptotic law for eigenvalues.*

It has long been known for a bounded domain D in the plane that the eigenvalues λ_n , $0 < \lambda_n \leq \lambda_{n+1}$, of the membrane problem are distributed according to Weyl's law $N(t) = \sum_{\lambda_n \leq t} 1 = (\mu_2(D)/4\pi)t + o(t)$ as $t \rightarrow +\infty$. In a recent paper (Comm. Sem. Math. Univ. Lund, suppl. (Riesz) (1952) p. 177) Pleijel gets over $\omega \geq 1$ the estimate

$\sum_{n=1}^{\infty} 1/\lambda_n(\lambda_n + \omega) = (\mu_2(D)/4\pi)(\ln \omega/\omega) + (C/\omega) + (l(B)/8)(1/\omega^{3/2}) + O(1/\omega^2)$ for an infinitely differentiable boundary B , $l(B)$ being the total length of B . This estimate is much more detailed than the usual one from which Weyl's law follows by Tauberian theorems, and Pleijel suggests that by using his methods to extend his estimate over a cone of complex ω it should be possible to get a more precise form of Weyl's law. We show that from Pleijel's estimate alone follows $N(t) = (\mu_2(D)/4\pi)t - (l(B)/4\pi)t^{1/2} + \tilde{O}(\ln t)$ as $t \rightarrow +\infty$, where precisely the averaged \tilde{O} estimate says that $F(t) = N(t) - \{(\mu_2(D)/4\pi)t - (l(B)/4\pi)t^{1/2}\}$ has $|\int_{\lambda_1}^{\infty} \exp(-\rho^2/2)(\ln u/t^2)dF(t)| \leq M_{\rho} \ln u$ over $u \geq e$ for every $\rho > 0$. Furthermore, any ordinary asymptotic series for $N(t)$ must agree with this \tilde{O} type one term for term as far as the ordinary series goes. This consistency result proves false a conjecture of Minakshisundaram (*Symposium on spectral theory*, Stillwater, Okla., 1951, p. 331, no. 2). (Received October 19, 1953.)

25t. R. B. Davis: *A special case of the normal derivative problem for a composite third-order linear partial differential equation.*

The normal derivative problem for a third-order composite partial differential equation consists in prescribing u around the boundary of a region, and $\partial u/\partial n$ around part of this boundary. This problem is solved for the equation $\partial \Delta u/\partial x = \Delta(\partial u/\partial x) = 0$, where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, for a special type of convex region, bounded in part by a straight line $x = \text{constant}$. Both existence and uniqueness are proved. Smoothness of bounding arcs, and of boundary values, is assumed in a certain sense, although the region contains corners. The method involves solution first of a Neumann problem for $\Delta u = 0$, by which the given problem is made symmetric and is reduced to a different type of boundary value problem considered in O. Sjöstrand, *Arkiv för Matematik, Astronomi och Fysik* vol. 26A (1939) pp. 1-10. The hypothesis includes certain restrictions which are necessary to the process of symmetrization. (Received September 8, 1953.)

26. D. J. Dickinson: *On Bessel and Lommel polynomials.*

The modified Lommel polynomials $h_{n,\nu}(x)$ are defined in terms of the Lommel polynomials (see Watson's *Bessel functions*, §9.6) by $h_{n,\nu}(x) = R_{n,\nu}(1/x)$. The $h_{n,\nu}(x)$, for $\nu > 0$, are orthogonal polynomials with respect to the complex weight function $J_{\nu}(1/x)/J_{\nu-1}(1/x)$ where the integration is taken over a circle in the complex plane that includes the finite singularities of the weight function. Also, let $j_{\nu-1,n}$ be the n th positive zero of the Bessel function $J_{\nu-1}(x)$, $\nu > 0$, and for $n < 1$, let $j_{\nu-1,n} = -j_{\nu-1,-n+1}$. Then for $\nu > 0$, $\int_a^b h_{m,\nu}(x)h_{n,\nu}(x)d\alpha_{\nu}(x) = \delta_{m,n}/2(\nu-n)$ where the real interval $[a, b]$ is $[-j_{\nu-1,1}^{-1}, +j_{\nu-1,1}^{-1}]$. The $d\alpha_{\nu}(x)$ is a Stieltjes distribution where $\alpha_{\nu}(x)$ is a nondecreasing step function with a jump of $j_{\nu-1,n}^{-2}$ at the point $x = j_{\nu-1,n}^{-1}$ for all integral n , positive, negative, and zero. One relation between the Bessel polynomials $y_n(x)$ (see Krall and Frink, *Trans. Amer. Math. Soc.* vol. 65 (1949) pp. 100-115) and the modified Lommel polynomials is $i^n y_n(-ix) = h_{n,1/2}(x) + ih_{n-1,3/2}(x)$. From this relation and a known property of the Lommel polynomials, it is deduced that the absolute value of the zeros of $y_n(x)$ have an upper bound that approaches zero as n increases. From similar relations between the Bessel polynomials and the modified Lommel polynomials, one may deduce new facts about one polynomial set from known facts about the other. (Received September 15, 1953.)

27. Michael Fekete: *Asymptotic behavior of certain polynomials of least deviation on point-sets of a given transfinite diameter.*

Given an infinite compact point set of the z -plane, associate to every polynomial $A_n(z) = z^n + \sum_{\nu=1}^n a_\nu z^{n-\nu}$ of degree $n \geq 1$ with leading coefficient 1, in an arbitrary but definite way, a non-negative finite number $D = D(A_n(z), S)$ as its *deviation* on S (from 0). Call D a *monotone* deviation if D decreases when $A_n(z)$ is replaced by its *underpolynomials* $B_n(z) = z^n + \sum_{\nu=1}^n b_\nu z^{n-\nu}$ on S (if such there be) of the same degree and with the same leading coefficient which satisfy $|B_n(z)| < |A_n(z)|$ whenever $z \in S$ except the common roots $z \in S$ of $A_n(z) = 0, B_n(z) = 0$ if there are such. Call D a *continuous* deviation if $|D(A_n(z), S) - D(B_n(z), S)| \leq \epsilon$ whenever $|A_n(z) - B_n(z)| \leq \delta(\epsilon)$ for all $z \in S$. Monotony and continuity of D ensure the existence of a *polynomial of least deviation* $A_n^*(z; S, D) = z^n + \sum_{\nu=1}^n a_\nu^* z^{n-\nu}$ on S of degree n with respect to D . The *Chebyshev deviation* $D = \text{Max}_{z \in S} |A_n(z)|$ and the *Bessel-deviation* $D = \int_S |A_n(z)|^2 ds$ (where S is a rectifiable Jordan arc or Jordan contour) have both aforesaid properties thus possess polynomials $A_n^*(z)$ of least deviation. The author (Math. Zeit. (1923)) and G. Szegő (Math. Zeit. (1921)) have respectively shown: the pertaining *least deviations* $\delta_n = D(A_n^*(z), S)$, in both cases, satisfy $\lim_{n \rightarrow \infty} \delta_n^{1/n} = \tau(S)$ = the transfinite diameter of S ; S being, however, in the second case, *analytic*. Recent allied investigations of P. Davis and H. Pollak (Proc. Amer. Math. Soc. (1952)) induced author to include all mentioned results in one general proposition, that is: Let $D = D(A_n(z), S)$ be a monotone and continuous deviation on S and $A_n^*(z)$ a polynomial of least deviation on S w. r. to D . Put $\delta_n = D(A_n^*(z), S)$, $\mu_n = \text{Max}_{z \in S} |A_n^*(z)|$. Then $\lim_{n \rightarrow \infty} \delta_n^{1/n} = \lim_{n \rightarrow \infty} \mu_n^{1/n} = \tau(S)$ provided there is a *supermultiplicative Hausdorff function* $h(r)$ [continuous and strictly increasing with r , subject to $h(0) = 0, h(1) = 1$], such that the ratio $h(D(A_n(z), S) : h(\text{Max}_{z \in S} |A_n(z)|))$ admits, for all $n \geq 1$, an upper bound $U(S)$ and, for $n \geq n_0(\epsilon)$, a lower bound $L_n(S, \epsilon)$ as well, satisfying $\lim_{n \rightarrow \infty} (L_n(S, \epsilon))^{1/n} = 1 - \epsilon$. Inclusion of the particular results of Davis and Pollak alluded to follows from the *semicontinuity* of $\tau(S)$ *from below* in some noteworthy special cases (to be published later). (Received November 2, 1953.)

28t. A. O. Huber: *The reflection principle for polyharmonic functions.*

Let G denote a region of R_n , the boundary of which contains an open subset S of $x_1 = 0$. If the function w is p -harmonic in G and if w/x_1^{p-1} assumes the boundary value 0 on S , then w can be continued analytically beyond S into the reflected domain by putting $w(-x_1, x_2, \dots, x_n) = (-1)^p \sum_{k=0}^{p-1} A_k x_1^{p+k} \Delta^k [w(x_1, x_2, \dots, x_n)/x_1^{p-k}]$. The A_k are independent of p and defined by: $A_k = [(-1)^k - \sum_{\nu=0}^{k-1} A_\nu (k+\nu)! / (k-\nu)!] / (2k)!$ ($k = 1, 2, 3, \dots$), $A_0 = 1$. This result is known for $p = 1$ and $p = 2$ (for the latter case see R. J. Duffin, Bull. Amer. Math. Soc. Abstract 59-4-358). This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 4, 1953.)

29t. Raphael M. Robinson: *Extremal problems for star mappings.*

The maximum possible value of $\Re \{ B \log f'(z_0) + C \log [f(z_0)/z_0] \}$, where B and C are complex numbers (not both zero) and $0 < |z_0| < 1$, for the class of functions $f(z) = z + \dots$ which map $|z| < 1$ conformally onto star-shaped regions in a one-to-one way, can be attained only by a function $w = f(z)$ which maps $|z| < 1$ onto the w -plane with one or two radial slits. Examples are given of both alternatives. In particular, it is shown that the set of possible values of $f'(z_0)$ is the map of $|z| \leq |z_0|$ by $w = (1+z)/(1-z)^3$, at least if $|z_0| \leq 0.6$. (Received September 11, 1953.)

30. Arthur Sard: *Approximation and projection.*

Let H_Y be the usual space of (classes of) functions $u = u(y)$, $y \in Y$, absolute square integrable relative to a measure m on a space Y . Similarly let H_Ω be the space of functions $z = z(\omega)$, $\omega \in \Omega$, absolute square integrable relative to a probability p on Ω . Assume that H_Y and H_Ω are separable. Let $H_{Y\Omega}$ be the space of functions $v = v(y, \omega)$, $y \in Y$, $\omega \in \Omega$, absolute square integrable mp on $Y \times \Omega$. Then $H_{Y\Omega}$ is the Kronecker product of H_Y into H_Ω . Let H_X be any separable Hilbert space; let $H_{X\Omega}$ be the Kronecker product of H_X into H_Ω . Suppose given an element g of $H_{X\Omega}$, an element h of $H_{Y\Omega}$, and a linear set \mathcal{G} of bounded linear operators on H_X to H_Y . Efficient and strongly efficient operators T (relative to g, h, \mathcal{G}) are defined in a natural way with a view to the approximation of h by Tg , $T \in \mathcal{G}$. Operators arbitrarily close to efficiency always exist; they are characterized. Necessary and sufficient conditions for strong efficiency and nearly strong efficiency are given. In the case in which the measure m is atomic, a necessary and sufficient condition that efficiency imply strong efficiency is given. The possibility of an advantageous change of topology in H_X is discussed. (Received August 18, 1953.)

31. I. E. Segal: *Direct limits of probability spaces.*

A theorem of Kolmogoroff's, that there exist random variables having preassigned joint distributions provided these satisfy the obviously necessary consistency conditions, is extended from the case of real-valued to that of general-valued random variables. The required variables are defined as set transformations and exhibited essentially as functions on a space that is a direct limit of probability spaces associated with the given distributions, and such limits are examined in general. The noncommutative analogue of such direct limits and also of conditional probability are treated at the same time, and noncommutative extensions obtained for some known martingale convergence theorems. The results are pertinent to the algebraic isomorphism problem for finite rings of operators. (Received September 4, 1953.)

32. V. L. Shapiro: *Subharmonic functions of order r .*

$F(x, y)$ is said to be subharmonic of order r in a domain G if $F(x, y)$ is in class $C^{2(r-1)}$ and $\Delta^{r-1}F(x, y)$ is subharmonic in G (Δ^r standing for the Laplacian operator iterated r times and coinciding with the identity operator when $r=0$). $F(x, y)$ is said to have a generalized r th Laplacian of the second kind at (x_0, y_0) equal to a_r if $\pi^{-1}t^{-2} \int_0^{2\pi} u du \int_0^{2\pi} F(x_0 + u \cos \theta, y_0 + u \sin \theta) d\theta = a_0 + a_1 t^2 / 2(2!)^2 + \dots + a_r t^{2r} / (r+1)(2^r r!)^2 + o(t^{2r})$ where $t > 0$ and a_i ($i=0, \dots, r$) are constants. Designating this generalized r th Laplacian by $\Delta_2^r F(x_0, y_0)$, the following theorem is proved: Let $F(x, y)$ be in class $C^{2(r-1)}$ in a domain G and let E be a bounded closed set of capacity zero contained in G . Suppose $\Delta_2^r F(x, y)$ is non-negative for (x, y) in $G-E$. Then $F(x, y)$ is subharmonic of order r in G . (Received August 10, 1953.)

33t. Milton Sobel: *On a generalization of an inequality of Hardy, Littlewood, and Pólya.*

Let $A = \{a_1 \leq a_2 \leq \dots \leq a_n\}$, $B = \{b_1 \leq b_2 \leq \dots \leq b_n\}$ denote sets of real numbers. Let (B_1, B_2) denote any of the C_n^m partitions of B with n_1 elements in B_1 and $n_2 = n - n_1$ elements in B_2 . Let $A_1^* = \{a_\alpha\}$, $B_1^* = \{b_\alpha\}$, $A_2^* = \{a_\beta\}$, $B_2^* = \{b_\beta\}$ ($\alpha = 1, 2, \dots, n_1$; $\beta = n_1 + 1, \dots, n$). For each partition (B_1, B_2) consider the $n_1! n_2!$ cross-products $v_{pq} = \sum_{i=1}^{n_1} a_i b_{j_i}$ ($p = 1, 2, \dots, C_n^m$; $q = 1, 2, \dots, n_1! n_2!$) formed by associating B_i with A_i^* ($i = 1, 2$). Let v_q^* denote the $n_1! n_2!$ crossproducts formed by associating B_i^* with A_i^* ($i = 1, 2$). Then for each partition p there is a one-to-one pairing of the

v_q^* and $v_{p,q}$ such that $v_q^* \geq v_{p,q}$ ($q=1, 2, \dots, n_1! n_2!$). A generalization of this result in which B is partitioned into (B_1, B_2, \dots, B_k) with n_i elements in B_i also holds. The case $n_i=1$ ($i=1, 2, \dots, k$) reduces to the result: For each p . ($p=1, 2, \dots, n!$), $\sum_{i=1}^n a_i b_i = v^* \geq v_p = \sum_{i=1}^n a_i b_{i,i}$, given in *Inequalities*, p. 261, by Hardy, Littlewood, and Pólya. Research sponsored by ARDC. (Received August 24, 1933.)

34t. J. S. Thale: *Univalence of continued fractions and Stieltjes transforms.*

Using only the definition and some elementary properties of univalent functions, the author obtains first some theorems on the univalence of functions of the form $\int d\phi(t)/(1+xt)$, $\int x d\phi(t)/(1+xt)$ with suitable upper and lower limits for the Stieltjes integrals. These theorems also apply to certain classes of continued fractions associated with such Stieltjes transforms. Further results on univalence and starlike character of additional classes of continued fractions are obtained by means of an iterative process based on value region results for continued fractions. A typical result is: If $|a_p| \leq 1/4$, then the function $f(x) = x/1 + a_1 x/1 + a_2 x/1 + \dots$ is univalent for $|x| < 4(2^{1/2}/(3+2(2^{1/2})))$ and is starlike with respect to the origin for $|x| \leq 8/9$. (Received September 8, 1953.)

35t. Louis Weisner: *Group-theoretic origin of certain generating functions.*

Let $L(x, d/dx, n)v=0$ be a linear ordinary differential equation containing a parameter n , and suppose the coefficients, as functions of n , are rational integral functions. Substituting $y\partial/\partial y$ for n , the equation is converted into a linear partial differential equation $L(x, \partial/\partial x, y\partial/\partial y)u=0$. If $v=v_n(x)$ is a solution of the ordinary differential equation, then $u=y^n v_n(x)$ is a solution of the corresponding partial differential equation; and conversely. There are four types of linear partial differential equations in two variables, of the second order, that admit a three-parameter Lie group and one that admits a five-parameter group. The group is used to obtain solutions of the partial differential equation. These solutions provide generating functions for solutions of the corresponding ordinary differential equations. In this way a large number of generating functions are obtained systematically for certain classes of hypergeometric, and related, functions. (Received September 1, 1953.)

APPLIED MATHEMATICS

36t. Lipman Bers: *Existence and uniqueness of a subsonic gas flow past a given profile. II.*

A new, much simpler, proof is given for the gas-dynamical existence theorem announced in the first abstract with the same title (Bull. Amer. Math. Soc. Abstract 59-4-402). The new proof does not use the Schauder-Leray degree; it is an application of a general method of estimating solutions of boundary value problems for quasi-linear elliptic equations in the plane (see abstract of paper *An application of function theory of quasi-linear elliptic equations.*) (Received September 9, 1953.)

37. Gabriel Horvay (p) and F. N. Spiess: *Orthogonal edge polynomials in the solution of boundary value problems.*

The paper, now being published in the Quarterly of Applied Mathematics, generalizes the variational method of Kantorovitch-Poritsky by using in the expansion

$\phi(x, y) = \sum c_n g_n(x) f_n(x, y)$ of the unknown function ϕ , an orthonormal set of boundary polynomials $f_n(x, y)$ which are complete with respect to the prescribed boundary conditions. The factors $g_n(x)$ are determined from Euler-Lagrange equations. (Received August 27, 1953.)

38. Martin Krakowski and Abraham Charnes (p): *Stokes' paradox and biharmonic flows.*

A generalized form of Stokes' paradox is exhibited as a consequence of a new uniqueness theorem for biharmonic flows in the complement of compact, possibly multiply connected, plane regions allowing $o(\log r)$ velocity growth at infinity. This answers a question of G. Birkhoff (*Hydrodynamics*, 1950). The single-valuedness of pressure and vorticity required by the flow interpretation precludes an additional possibility suggested by S. G. Mihlin (pp. 209-210 of *Integral equations and their applications*, OGIZ, 1949) from an elasticity interpretation. (Received September 24, 1953.)

39. I. F. Ritter: *Accelerated steepest descent methods. I.*

For real matrix A the *least squares solution* of the equation $Ax=y$ is found by *explicit* use of the system of *orthogonal vectors* forming the columns of a matrix $B=AT$. The triangular matrix T is built up simultaneously with B . Because $B'B$ is diagonal the solution $x=T(B'B)^{-1}(B'y)$ is easily computed. If A is square and nonsingular, then $A^{-1}=T(B'B)B'$. For any matrix A the residual vector $y-Ax$ has the minimum possible norm. Nontrivial solutions of $Ax=0$, if any, appear as columns of T corresponding to zero columns in B . These columns of T are multiplied by arbitrary parameters in forming x . Besides unrestricted applicability the procedure has the advantages of skipping the computation of the normal equations $A'Ax=A'y$ and of permitting a rapid correction of the error in x caused by rounding off or small mistakes. The approximate solution x^1 is improved by $x=x^1+T(B'B)^{-1}(B'r)$ with $r=y-Ax^1$ and this is iterated until the norm of $y-Ax$ assumes its minimum value. (Received November 2, 1953.)

40*t*. I. F. Ritter: *Accelerated steepest descent methods. II.*

Methods of steepest descent for solving the eigenvalue problem $Ay=\lambda By$, with matrices A and B , for eigenvalues λ and eigenvectors y [Hestenes and Karush, *Journal of Research, National Bureau of Standards* vol. 47 (1951) or Cooper, *Quarterly of Applied Mathematics* vol. 6 (1948)] maximize or minimize the Rayleigh quotient while minimizing the residual vector. The consequent restriction to Hermitian matrices can be removed if the Rayleigh quotient is replaced by the quotient $g(x) = (Ax \cdot \overline{Bx}) / (Bx \cdot \overline{Bx})$ of two Hermitian scalar products for the approximating vector x . With a suitable choice $x^{(0)}$ the sequence of approximations appears as $x^{(i+1)} = x^{(i)} + a_i r^{(i)}$, with $r^{(i)} = Ax^{(i)} - g(x^{(i)})Bx^{(i)}$, $i=0, 1, 2, \dots$. The scalars a_i and $h_i = g(x^{(i+1)}) - g(x^{(i)})$ can be treated as independent variables in minimizing the residual norm $(r^{(i+1)} \cdot \overline{r^{(i+1)}})$, the a_i with smallest $|h_i|$ being readily found approximately from the resulting equations. With this a_i the sequence $x^{(i)}$ converges to an eigenvector y belonging to the eigenvalue $\lambda = g(y)$ nearest to $g(x^{(0)})$, without requiring repeated orthogonalizations. To accelerate the procedure the sub-routine of finding a_i is replaced by a simpler one using for a_i one root of the quadratic equation which eliminates h_i from those two equations of the system $r^{(i+1)}=0$ in which the numerically largest components of $r^{(i)}$ appear. As soon as h_i is sufficiently small the method given

in part I leads from a rough approximate eigenvalue g_i to a greatly improved eigenvector y as solution of $Ay = g_i By$. (Received November 2, 1953.)

GEOMETRY

41. A. P. Dempster and Seymour Schuster (p): *Constructions of poles and polar primes.*

Von Staudt, in 1847, introduced the idea of handling a symmetric polarity in projective space by means of a self-polar simplex and another pair of corresponding elements. Neither he nor anyone in the school of projective geometers which followed gave any explicit construction for the prime of an arbitrary point, or the pole of an arbitrary prime relative to a given polarity. H. S. M. Coxeter exhibited such constructions for two-space. In the present paper we give two methods for such constructions in n -space: one, which calls heavily upon the use of Hesse's Theorem (as does Coxeter's construction); and a second, which reduces the problem in n -space to 2^r constructions in $(n-r)$ -flats ($1 \leq r \leq (n-1)$). (Received September 10, 1953.)

42. A. O. Huber: *An isoperimetric inequality for surfaces of variables Gaussian curvature.*

Let C denote a rectifiable Jordan curve of length L enclosing on a surface of piecewise continuous curvature K a simply-connected piece of surface S of area A . Then the following inequality holds: $L^2 \geq 2A(2\pi - I)$, where $I = \iint_S K^+ dS$ with $K^+ = \max [K, 0]$. Equality holds if and only if $K = 0$ on S and C is a geodesic circle. For $K \leq 0$ we thus obtain a theorem of E. F. Beckenbach and T. Radó (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 662-674), and for $K \geq 0$ a result of F. Fiala (Comment. Math. Helv. vol. 13 (1940-41) pp. 293-346). The proof makes use of an inequality of T. Carleman (Math. Zeit. vol. 9 (1921) pp. 154-160), Hölder's inequality, and some properties of conformal mapping. This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 9, 1953.)

LOGIC AND FOUNDATIONS

43t. Theodore Hailperin: *Identity and description in first-order axiom systems.*

In first-order axiom systems quantifiers apply only to individual variables and not to predicates, which are assumed fixed for the system. In such a system the predicate identity can be defined in terms of the subject-matter predicates and need not be taken as a primitive undefined notion. Having identity in a system it then becomes possible to introduce descriptions ("the x such that Fx "), either as explainable in context or axiomatically. This paper gives, for an arbitrary first order system, complete sets of axioms for the essentially different versions of descriptions found in Hilbert-Bernays' *Grundlagen der Mathematik* and in Rosser's *Logic for mathematicians*. (Received August 25, 1953.)

STATISTICS AND PROBABILITY

44t. R. E. Bechhofer and Milton Sobel: *On a sequential ranking procedure.* Preliminary report.

Let X_{ij} be normally, independently distributed from population π_i with mean

μ_i^0 and unit variance ($i=1, 2, \dots, k; j=1, 2, \dots$). The k ranked means $\nu_1^0 < \dots < \nu_k^0$ are known, it being unknown which population has mean ν_r^0 ($r=1, 2, \dots, k$). Problem: Which population has mean ν_k^0 ? Let $X_i^m = \sum_{j=1}^m X_{ij}$; denote the ranked X_i^m by $Y_1^m < \dots < Y_k^m$. Let $S_m = [\sum \exp \{ \sum_{i=1}^{k-1} Y_i^m \nu_i + Y_k^m \nu_k \}] \div [\sum \exp \{ \sum_{i=1}^k Y_i^m \nu_i \}]$, (j_1, j_2, \dots, j_s) denoting a permutation of $(1, 2, \dots, s)$, the summations in the numerator and denominator being over all permutations of $(1, 2, \dots, k-1)$ and $(1, 2, \dots, k)$, respectively. For $1/k < \gamma < 1$, the sequential procedure P_γ below has probability of a correct choice $\geq \gamma$. Procedure P_γ : "At the m th stage ($m=1, 2, \dots$) observe (x_{1m}, \dots, x_{km}) . Stop if $S_m \geq \gamma$, and decide that the population yielding Y_k^m has mean ν_k^0 ; continue if $S_m < \gamma$." The probability of termination is unity. The case of unknown ranked means has also been treated. Similar procedures apply for selecting the t populations with largest parameters from k Koopman-Darmois distributions having the same form. Research sponsored by ARDC. (Received August 24, 1953.)

45t. Jack Kiefer and Jacob Wolfowitz: *On the theory of queues with many servers.*

Ergodic properties of the following stochastic process are studied: For $i \geq 1$, let $t_i \geq t_0 = 0$ be the time of arrival of the i th person at a system of $s \geq 1$ machines, where he waits his turn until a machine is available to serve him, say at time $t_i + w_i \geq t_i$. This machine is then occupied by him for time $R_i \geq 0$. Let $g_i = t_i - t_{i-1}$. $\{R_i\}$ and $\{g_i\}$ are independent sequences of identically distributed and independent chance variables. An s -dimensional random walk W_i , with w_i its first component, is studied. Let F_i (F_i^*) be the d.f. of W_i (w_i). It is shown that $F(x) = \lim_{i \rightarrow \infty} F_i(x)$ exists and satisfies a certain integral equation (I.E.); $F^*(x) = \lim_{i \rightarrow \infty} F_i^*(x)$ also exists. Put $\rho = ER_i/sEg_i$. Then F and F^* are d.f.'s if and only if $\rho < 1$, and F is then the unique d.f. solution to the I.E. Except in the trivial case where $P\{R_i = sg_i\} = 1$, if $\rho \geq 1$ then $F = 0 = F^*$, and the I.E. has no d.f. solution. Always $F^*(x) = F(x, \infty, \dots, \infty)$. Results on the limiting length of the line are also proved. (Received August 5, 1953.)

TOPOLOGY

46. R. D. Anderson and Mary-Elizabeth Hamstron (p): *A note on continuous collections of continuous curves filling up continuous curves in the plane.*

It is proved that if G is a continuous collection of nondegenerate continuous curves filling up a compact continuous curve in the plane, then with respect to its elements regarded as points G is either an arc or a simple closed curve. Considerable use is made of some of R. D. Anderson's results (*Continuous collections of continuous curves in the plane*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 647-657) particularly Theorem VI, which states that if G is a compact continuous curve with respect to its elements, then G is a hereditary continuous curve such that the closure of its set of emanation points is totally disconnected. Use is also made of Theorem I of that paper, from which it follows that if H is a closed subcollection of G , then it is not true that both the collection of arcs and the collection of simple closed curves of H are dense in H . (Received September 8, 1953.)

47. Lorenzo Calabi: *Topologies on sets of local mappings.*

\mathfrak{M} and \mathfrak{S} being two families of subsets of a set E , F a uniform space with entourages V , call: $\mathcal{L}_{\mathfrak{M}}(E, F)$ the set of the pairs (u, M) where $M \in \mathfrak{M}$ and u is a mapping

of M into F ; $(A, V)_u$ with $A \in \mathfrak{S}$, $A \subset M$, the set of local mappings $(v, N) \in \mathcal{L}_{\mathfrak{M}}(E, F)$ verifying $N \supset A$ and $(v(x), u(x)) \in V$ for $x \in A$. The sets $(A, V)_u$ are fundamental neighborhoods of (u, M) in the topological space $\mathcal{L}_{\mathfrak{M}}(E, F)$ provided \mathfrak{M} and \mathfrak{S} satisfy some simple conditions. $\mathcal{L}_{\mathfrak{M}}(E, F)$ is not separated, not uniformisable; under some assumptions on \mathfrak{M} , $\mathcal{L}_{\mathfrak{M}}(E, F)$ is connected and has no nonconstant continuous mappings into a separate space. Suppose E compact, metric space; \mathfrak{M} set of the open sets with finite many components; \mathfrak{S} set of the compact sets different from E . Put then $\mathcal{L}_{\mathfrak{M}}(E, F) = \mathcal{L}_0$. Assume $\mathcal{W} \subset \mathcal{L}_0$ to verify some of the properties of the set of analytic functions in the sense of Weierstrass. Call \mathcal{A}_0 the subspace of \mathcal{L}_0 whose elements are unions of restrictions of mappings in \mathcal{W} . \mathcal{A}_0 generalizes the space of local analytic functions of Fantappiè, is closed in \mathcal{L}_0 , connected and arcwise connected. If E and F have convenient algebraic structures, \mathcal{A}_0 is locally arcwise connected. The results apply in particular for complex and hypercomplex analytic functions. (Received September 3, 1953.)

48. H. D. Friedman: *Topologies on a space of continuous functions.*

Let S be the set of all real-valued continuous functions defined on the closed interval $[0, 1]$. Seven well known topologies for S are related—those resulting from: uniform, pointwise, bounded pointwise, and weak convergences; and pointwise, bounded pointwise, and weak neighborhood systems. All seven spaces are shown to be Hausdorff spaces, but in none of the seven is the unit sphere compact. The topologies are partially ordered under the relation “weaker than.” The most important result follows: Let $\{f_n(x)\} \subset S$, $f(x) \in S$. We say that $f_n(x)$ converges *weakly* to $f(x)$ provided that for every function $\alpha(x)$ of bounded variation ($\alpha(x)$ not necessarily in S) $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) d\alpha(x) = \int_0^1 f(x) d\alpha(x)$. Then for $f_n(x)$ to converge weakly to $f(x)$ it is necessary (and sufficient, but this is well known) that $\{f_n(x)\}$ be uniformly bounded and pointwise convergent to $f(x)$. (The above is an abstract of the writer’s doctoral dissertation. The writer is indebted to Professor Orrin Frink, of the Pennsylvania State College, for his advice and encouragement.) (Received September 10, 1953.)

49. I. S. Gál: *Relations between knot invariants.*

In a previous abstract [Bull. Amer. Math. Soc. Abstract 59-6-698] we defined a knot invariant Λ in terms of an abstract cycle associated with a regular normed projection of a knot K . Now we can prove the following results: Let $\Gamma = (v_{i,2} - v_{i,1}, v_{i,4} - v_{i,3}, \dots, v_{i,2h} - v_{i,2h-1})$ be the Seifert matrix where the v_{ik} ’s denote the crossing numbers. Let $\Delta(x) = \sum a_k \cosh kx$, where $(\dots, a_1, 2a_0, a_1, \dots)$ are the coefficients of the Alexander polynomial of K . Then we have $\Lambda/2 = \sum (v_{2i-1,2j-1} v_{2i,2j} - v_{2i-1,2j} v_{2i,2j-1}) = \Delta''(0) = \sum k^2 a_k$. Consequently if the Alexander polynomial of K is trivial, then $\Lambda = 0$, but not conversely. (Received September 10, 1953.)

50t. R. D. Johnson, Jr.: *An antipodal point theorem for mapping the 2-sphere into the reals.*

Let $f(S^2) = R_0$ be a (continuous) mapping of the 2-sphere into the reals. Then there exists a point r contained in R_0 such that some component of $f^{-1}(r)$ contains a pair of antipodal points of S^2 . The following result is obtained as a corollary to this theorem: Let A and B be two closed sets such that $A \cup B = S^2$. Then there exists a continuum K containing a pair of antipodal points of S^2 which is contained in one of the three sets $(A - B)$, $(B - A)$, and $(A \cap B)$. (Received September 15, 1953.)

51. Harold W. Kuhn: *Contractibility and convexity.*

The intersection of a set X in a euclidean space with a hyperplane of support is called a *face* of X . With this definition, the following purely geometrical result appears as a by-product of the theory of games. Theorem: If X is a contractible polyhedron with contractible faces, then X is a convex set. (Received September 9, 1953.)

52t. C. W. Saalfrank: *On the universal covering space and the fundamental group.*

Let X be an arcwise connected, Hausdorff space, and let X_p^* be the universal covering space associated with X . It is shown that if A is a retract of X , then A_p^* is homeomorphic to a retract of X_p^* . A subgroup F of a group G is called a retract of G provided there exists a homomorphism h such that $h(G) = F$ and such that $h(f) = f$ for all $f \in F$. It is shown that if A is a retract of X , then the fundamental group $\pi(A)$ is isomorphic to a retract of the fundamental group $\pi(X)$. (Received June 16, 1953.)

53. E. S. Wolk (p) and J. K. Goldhaber: *Maximal ideals in rings of bounded continuous functions.*

Let X be a topological space, A a topological ring, and $C^*(X, A)$ the ring of bounded (in the sense of Shafarevich, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 133–135, and Kaplansky, Amer. J. Math. vol. 69 (1947) pp. 153–183) continuous functions from X to A . On the basis of the definition of boundedness as given by Shafarevich and Kaplansky, a definition of X completely regular with respect to A is given. Now let Σ be the set of all infinite sequences of distinct points of X which have no limit point in X , and let S be a maximal sub-lattice in Σ . Define $M(S) = \{f \in C^*(X, A) \mid \lambda \in S \text{ implies that } f \rightarrow 0 \text{ on a subsequence of } \lambda\}$. The following theorems are proved: (1) Let A be a topological division ring of type V (Kaplansky, Duke Math. J. vol. 14 (1947) pp. 527–541) with continuous inverse and satisfying the first axiom of countability. Let X be a noncompact Hausdorff space which is completely regular with respect to A . Then $M(S)$ is a proper maximal free ideal in $C^*(X, A)$. (2) $C^*(X, A)/M(S)$ is isomorphic with A if and only if every bounded closed set in A is compact. The following question remains open: What further conditions on A , if any, are required to insure that every proper maximal free ideal in $C^*(X, A)$ is of the above form? (Received August 24, 1953.)

L. W. COHEN,
Associate Secretary