self-contained, while on the other hand the student who wishes to pursue the subject further can do so.

A. W. GOODMAN

Bernstein polynomials. By G. G. Lorentz. (Mathematical Expositions, no. 8.) University of Toronto Press, 1953. 10+130 pp. $5.75.

For a function $f(x)$ defined for $0 \leqq x \leqq 1$ the formula

$$B_n(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

defines the Bernstein polynomial of order n of $f(x)$. Evidently $B_n(x)$ is a polynomial in x of order less than or equal to n. S. Bernstein, who introduced these polynomials, proved that if $f(x)$ is continuous for $0 \leqq x \leqq 1$, then

$$\lim_{n\to\infty} B_n(x) = f(x)$$

uniformly for $0 \leqq x \leqq 1$, thus obtaining a particularly simple and elegant proof of the Weierstrass approximation theorem. Bernstein polynomials are of interest in themselves, and in addition play an important role in several other mathematical topics, notably in the finite moment problem and in the related theory of Hausdorff summability.

Chapter I deals with the approximation of continuous functions by Bernstein polynomials. A great many results are proved of which the following may serve as an example: if $f(x)$ satisfies a Lipschitz condition of order α, $0<\alpha<1$, then

$$|f(x) - B_n(x)| = O(n^{-\alpha/2}) \qquad (n \to \infty),$$

uniformly for $0 \leqq x \leqq 1$. Care is taken to explain the relation of these results to the general theory of polynomial approximation.

Chapter II treats the changes which are necessary if $f(x)$ is no longer continuous, but merely integrable, or even only measurable. For example if $f(x)$ is integrable $B_n(x)$ may be replaced by

$$P_n(x) = \sum_{\nu=0}^{n}\binom{n}{\nu} x^\nu(1-x)^{n-\nu}\left[(n+1)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(t)dt\right].$$

It is shown that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

almost everywhere for $0 \leqq x \leqq 1$. Also considered are a number of modifications and generalizations of Bernstein polynomials. For example, if the basic interval is $(0, b)$ rather than $(0, 1)$ so that $B_n(x)$ is replaced by

$$B_n(x, b) = \sum_{\nu=0}^{n} f\left(\frac{b_\nu}{n}\right)\binom{n}{\nu}\left(\frac{x}{b}\right)^{\nu}\left(1 - \frac{x}{b}\right)^{n-\nu}$$

and if $f(x)$ is continuous and bounded for $0 \leqq x < \infty$, then

$$\lim_{n\to\infty} B_n(x, b_n) = f(x) \qquad (0 \leqq x < \infty)$$

if $b_n \to \infty$, $b_n = o(n)$, as $n \to \infty$. Also contained in this chapter are certain results on summability.

Chapter III is devoted to the finite moment problem. A sequence $\{\mu_k\}_0^\infty$ is said to be a moment sequence if there exists a function $\alpha(x)$ of bounded variation in $0 \leqq x \leqq 1$ such that

$$\mu_k = \int_0^1 x^k d\alpha(x).$$

The "moment problem" is to determine necessary and sufficient conditions that a given sequence be a moment sequence with $\alpha(x)$ of some prescribed form. Such conditions are most simply expressed in terms of the polynomial operator $\mathfrak{M}$ defined by

$$\mathfrak{M}[c_0 + c_1 x + \cdots + c_n x^n] = c_0\mu_0 + c_1\mu_1 + \cdots + c_n\mu_n.$$

For example $\{\mu_k\}_0^\infty$ is a moment sequence with $\alpha(x)$ nondecreasing if and only if

$$\mathfrak{M}\left[\binom{n}{\nu} x^\nu (1 - x)^{n-\nu}\right] \geqq 0 \qquad (\nu = 0, \cdots, n; n = 0, 1, \cdots).$$

Other theorems prescribe necessary and sufficient conditions that $\{\mu_k\}_0^\infty$ be a moment sequence with $\alpha(x) = \int_0^x a(u)du$, where $a(u)$ belongs to some linear space of functions over $0 \leqq u \leqq 1$. In this connection the author introduces the Banach spaces $\Lambda(\phi, p)$ and $M(\phi, p)$. Here $1 < p$, and $\phi(x)$ is a positive nonincreasing function normalized by the condition $\int_0^1 \phi(t)dt = 1$. A function $f(x)$ $(0 \leqq x \leqq 1)$ belongs to $\Lambda(\phi, p)$ if $\|f\|_\Lambda$ is finite where

$$\|f\|_\Lambda = \underset{\phi^\#}{\text{l.u.b.}} \left\{\int_0^1 |f(x)|^p \phi^\#(x)dx\right\}^{1/p},$$

$\phi^\#$ ranging over all measurable rearrangements of $\phi(x)$. Similarly

$f(x)$ belongs to $M(\phi, p)$ only if $\|f\|_M$ is finite where

$$\|f\|_M = \underset{e}{\text{l.u.b.}} \left\{ \int_e |f(x)|^p dx \Big/ \int_0^{me} \phi(x)dx \right\}^{1/p},$$

e ranging over all measurable subsets of $(0 \leqq x \leqq 1)$ for which $me > 0$, where me is the measure of e. Many properties are established for these spaces. The chapter concludes with a discussion of Hausdorff summability.

Chapter IV presents the quite difficult theory of the behavior of Bernstein polynomials in the complex domain. The results here are too complicated to admit simple description or illustration.

The exposition of the theory of Bernstein polynomials which this volume contains is quite complete. In collecting together this material, much of it from widely scattered and not easily available sources, the author has performed a valuable service.

I. I. HIRSCHMAN, JR.

An introduction to abstract harmonic analysis. By L. H. Loomis. New York, van Nostrand, 1953. 10+190 pp. $5.00.

Abstract harmonic analysis is a branch of mathematics based on the concept of the Fourier-Lebesgue integral transform on the real line, replacing that line by a locally compact topological group G, and the functions $e^{i\lambda x}$ by functions arising in representations of G (e.g. characters). The appeal of this rather new and growing doctrine is largely derived from its providing a field of application for concepts and techniques (in point set topology, abstract integration, linear spaces, operators in Hilbert space, and so forth) which had not yet found applications commensurate with their purely aesthetic value.

In any case, this doctrine provides an eagerly desired framework on which may be fastened, for contemplative or expository purposes, many of the more striking ideas of spectral theory and functional analysis. The structural members of this framework are more or less purely algebraic concepts from the theory of groups and their representations, algebras and their ideal theory.

The present work (an outgrowth of courses given at Harvard by G. W. Mackey and later the author), in tacit agreement with this point of view, is built around one of the simplest and yet most valuable topologico-algebraic concepts: the commutative Banach algebra.

The first chapter contains point set theoretic preliminaries, topological products of compact spaces, and Stone's generalization of