THE OCTOBER MEETING IN CAMBRIDGE

The five hundred sixth meeting of the American Mathematical Society was held at the Massachusetts Institute of Technology on Saturday, October 30. There were about 250 persons in attendance including 209 members of the Society.

An address entitled *On the Lebesgue-Stieltjes integral* was presented at the General Session by Professor Arne Beurling of The Institute for Advanced Study by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor B. O. Koopman presided.

Sessions for contributed papers were held in the morning and afternoon. Professors Eugene Lukacs, J. H. Roberts and Norbert Wiener presided at the morning sessions. Professors J. J. Gergen and R. M. Thrall presided at the afternoon sessions.

Abstracts of the papers presented follow. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Those papers with "*t*" following their numbers were presented by title. Mr. Mishael Zadek was introduced by Professor Garrett Birkhoff, Mr. D. S. Carter by Professor Joseph Lehner, and Mr. E. H. Brown, Jr., by Professor G. W. Whitehead.

ALGEBRA AND THEORY OF NUMBERS

1. R. A. Beaumont: *On the construction of R-modules.*

Let $S_1, S_2, \cdots, S_k$ be ideals (not necessarily distinct) in an integral domain R with characteristic zero and let G be the direct sum $S_1 \oplus S_2 \oplus \cdots \oplus S_k$ of the additive groups of these ideals. A mapping f of $R \times G$ into G is given by a set of mappings f_i, $i=1, 2, \cdots, k$, of $R \times S_1 \times S_2 \times \cdots \times S_k$ into S_i where $f(r, g) = f(r; s_1, s_2, \cdots, s_k) = (f_1(r; s_1, s_2, \cdots, s_k), \cdots, f_k(r; s_1, s_2, \cdots, s_k))$ for $r \in R$ and $g = (s_1, s_2, \cdots, s_k) \in G$. The following theorem is proved: Let f_i, $i=1, 2, \cdots, k$, be a polynomial function with coefficients in S_i. Then G is a left R-module with respect to the operator multiplication defined by the f_i if, and only if, each f_i is defined by $f_i(r; s_1, s_2, \cdots, s_k) = \sum_{j=1}^{k} a_j^{(i)} r s_j$, $a_j^{(i)} \in S_i$, and the matrix $A = (a_j^{(i)})$, $i, j = 1, 2, \cdots, k$, is idempotent. (Received September 16, 1954.)

2*t*. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons: I, II.*

A PT has $u^2+z^2=v^2+x^2=t^2$, $u^2-x^2=v^2-z^2=y^2$; $u, x, y, v, z, y, t=il \pm jk, ik \pm jl, 2(ijkl)^{1/2}, [(i^2+j^2)(k^2+l^2)]^{1/2}$. Dickson's *History* II, p. 632, Desboves[89] (6) has a misprint: the exponent 2 in $Zx'^4z'^2$ should be deleted. Adapting this to PT, the transforms D_{I} and D_{II} are: $i, j; k, l \to (kz^2-lw)^2/i$, $(lz^2-kw)^2/j$; $(iz^2-jw)^2/k$, $(jz^2-iw)^2/l$, and $(lx^2-kw)^2/i$, $(kx^2-lw)^2/j$; $(jx^2-iw)^2/k$, $(ix^2-jw)^2/l$, where $w=uv+ty$, and i, j and k, l correspond to Desboves' x^2, y^2. Thus 169, 4; 4, $1 \to 262238^2, 220169^2$; $859^2, 1186^2$, and $12134^2, 2297^2$; $43^2, 38^2$. Since i, j and k, l and their transforms mutually tetradize, each D_{I}, D_{II} pair yields 14 new PT. Thus the tetrad 43^2, 38^2; 4, l gives the 18th known sub-miniature PT ($t<10^4$): 7625, 3927, 8840, 5952, 6536, 9673. A. Cunningham (ibid. p. 634) "listed all $a^4+b^4=c\square<10^7$." But $43^4+38^4=17\cdot 569^2$, not in his list. Put $2z[wij/(k^2+l^2)]^{1/2}=r$, $2z[wkl/(i^2+j^2)]^{1/2}=s$, then $i^2+j^2\to(i^2+j^2)[(kz^2+lw+js)^2/i^2+s^2]$, under D_{I}, and $k^2+l^2\to[(iz^2+jw+lr)^2/k^2+r^2]\cdot(k^2+l^2)$. There are analogous transforms $\Delta_{\mathrm{I,II}}$ for the parameters such that t, u, v, y, z, $x=\iota\kappa\pm\theta\lambda$, $\iota\lambda\pm\theta\kappa$, $2(\iota\theta\kappa\lambda)^{1/2}$, $[(\iota^2-\theta^2)(\kappa^2-\lambda^2)]^{1/2}$, viz. ι, θ; κ, $\lambda\to(\kappa t^2+\lambda\omega)^2/\iota$, $(\lambda t^2-\kappa\omega)^2/\theta$; $(\iota t^2+\theta\omega)^2/\kappa$, $(\theta t^2-\iota\omega)^2/\lambda$ etc., $\omega=uv+xz$. Thus 49, 32; 9, $8\to 31^2$, $8\cdot 2^4$; $9\cdot 11^4$, $8\cdot 19^2$, tetradizable also. The Lenhart[45] PT (ibid. p. 506–507) is a transform of the Desboves[82] PT (p. 632); misprint, $c=81a^3x^6-\cdots$ should be $81a^3x^8$. A third misprint on this page is in Pepin[87]: the last term should be by^2 instead of bz^2. (Received September 14, 1954.)

3t. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons*: III, IV. Preliminary report.

A PT has u, x, v, z, y, $t=ef(g^2\pm h^2)$, $gh(e^2\pm f^2)$, $2efgh$, $[(e^2g^2+f^2h^2)(e^2h^2+f^2g^2)]^{1/2}$. Desboves[123] (Dickson's *History* II, p. 637) gave a transform on the general binary quartic without odd powers $=c\square$. Changing the notation, and adapting to the above quartic $=t^2$, one gets the transform D_{III}: $e, f, g, h, t\to(t^2y^2f^2-v^2z^2e^2)/e$, $(t^2y^2e^2-v^2z^2f^2)/f$, g, h, $t\{4u^2v^2x^2z^2-(u^2v^2+t^2z^2)^2\}$. Similarly, under D_{IV}: $e, f, g, h, t\to e, f$, $(t^2y^2h^2-u^2x^2g^2)/g$ $(t^2y^2g^2-u^2x^2h^2)/h$, $t\{(2uvxz)^2-(u^2v^2+t^2x^2)\}$. Cancelling g^2h^2, under D_{III} e, $f\to e\{4f^4t^2-vz(e^4-f^4)\}$, $f\{4e^4t^2-vz(e^4-f^4)\}$, so each magnitude of a PT divides its transform. In a Carmichael vector, the above factors of t^2 are $\boxed{S}$. The smallest example has e, f, g, h, $t=10$, 6, 7, 4, $1073\to 10\cdot 35254193$, $6\cdot 661577169$, 7, 4, $(29\cdot 479689681)(37\cdot 217141681)$ under D_{III}, which in this instance transforms a CV into a CV. In terms of the Lebesgue transforms, this Bulletin vol. 59 (1953) p. 148, $t\to(X_\Lambda^2u^2-T_\Lambda^2t^2)/t$ or $(Z_L^2v^2-T_L^2t^2)/t$ under D_{III} or D_{IV}. A PT also has v, y, t, u, z, $x=\epsilon\zeta(\eta^2\pm\theta^2)$, $\eta\theta(\epsilon^2\pm\zeta^2)$, $2\epsilon\zeta\eta\theta$, $[(\epsilon^2\eta^2-\zeta^2\theta^2)(\epsilon^2\theta^2-\zeta^2\eta^2)]^{1/2}$. There are analogous transforms $\Delta_{\mathrm{III,IV}}$, transliterating as ordered. (Received September 14, 1954.)

4t. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons*: V. Preliminary report.

The Desboves[79] (4), the Desboves[123] ($d=0$), the Carmichael[103], and Rignaux[106] transforms on quartics are all the same (refs. to Dickson's *History* II, chap. XXII). The identities are $I=i\{4i^4-3(i^2+j^2)^2\}^2=i\{(i^2-j^2)^2-4j^2(i^2+j^2)\}^2=i(i^4-6i^2j^2-3j^4)^2$. They underlie the PT of Euler[304] (ii), which is no. V of Rignaux, L'Int. des Math. vol. 26 (1919) pp. 55–57. The transform D_{V} on PT is i, j; k, $l\to I$ above, $J=j(3i^4+6i^2j^2-j^4)^2$; $K=k(k^4-6k^2l^2-3l^4)^2$, $L=l(3k^4+6k^2l^2-3l^4)^2$. Then $t\to T=t\prod_{i,j;k,l}(m^8+28m^6n^2+6m^4n^4+28m^2n^6+n^8)$, and $y\to Y=y\prod_{i,j;k,l}(m^4-6m^2n^2-3n^4)(3m^4+6m^2n^2-n^4)$. If T' and T'' are the homogeneous tetrads between t and T, that is the longest hypotenuses of PT with parameters i, j; I, J and k, l; K, L, then $T=T'T''/t$; similarly $Y=Y'Y''/y$. Also x, z, u, v divide their respective transforms X, Z, U, V. Simplest example: 13^2, 4; 4, $1\to 13^2\cdot 812988097^2$, $4\cdot 2449933763^2$; $4\cdot 157^2$, 863^2. Under the dual transform Δ_{V}: ι, θ, κ, $\lambda\to\iota(\iota^4+6\iota^2\theta^2-3\theta^4)^2$, etc. and $x\to X=X'X''/x=x\prod_{\iota,\theta;\kappa,\lambda}(\mu^8-28\mu^6\nu^2+6\mu^4\nu^4-28\mu^2\nu^6+\nu^8)$, $z\to Z=Z'Z''/z=z\prod_{\iota,\theta;\kappa,\lambda}(\mu^4+6\mu^2\nu^2-3\nu^4)(3\mu^4-6\mu^2\nu^2-\nu^4)$. A Carmichael vector depends on $abcd(a^2-b^2)(c^2-d^2)=\square=y^2/16$. Call a Carmichael transform C_{V}: a, b; c, $d\to a(a^4+6a^2b^2-3b^4)^2$, $b(3a^4-6a^2b^2-b^4)^2$; etc. Then applied to a given CV, $C_{\mathrm{V}}=P\Delta_{\mathrm{V}}$, where P denotes a Petrus transform. Simplest example: 5, 2; 6, $1\to 5\cdot 1177^2$, $2\cdot 1259^2$; $6\cdot 1509^2$, 3671^2. (Received September 14, 1954.)

5t. Eckford Cohen: *The quadratic singular sum.*

Let A be an arbitrary *odd* ideal of a finite extension F of the rational field. Denote by $\nu_s(\rho)$ the number of solutions of the congruence $\rho\equiv\alpha_1X_1^2+\cdots+\alpha_sX_s^2 \pmod{A}$, where $\rho, \alpha_1, \cdots, \alpha_s$ are integers of F, $(\alpha_i, A)=1$, $s\geqq 1$. In this paper, the function $\nu_s(\rho)$ is expressed in terms of a "singular sum" involving Hecke exponentials, and is then evaluated completely in terms of numerical functions over F. A number of special results are deduced; for example, if A is square-free, then $\nu_{2m+1}(\rho, A) = N^{2m}(A)\sum_{D|A}(-\alpha\rho/D)1/N^m(D)$, where $N(D)$ denotes the norm of D, (β/D) is the Jacobi symbol in F, and $\alpha=(-1)^{m+1}\alpha_1\cdots\alpha_{2m+1}$. The results obtained generalize theorems proved by the author in the rational case (Duke Math. J. vol. 21 (1954) pp. 9–28) and in the case of congruences to a prime-power ideal modulus (Trans. Amer. Math. Soc. vol. 75 (1953) pp. 444–470). (Received September 16, 1954.)

6. C. W. Curtis: *Lie algebras of algebraic linear transformations.*

Unless otherwise stated, it is assumed that all fields considered have characteristic zero. It is proved first that every locally algebraic linear transformation (l.t.) X in a vector space M of countable dimension over a perfect field K can be expressed uniquely in the form $S+N$, where S is locally algebraic and semi-simple, and N is locally nilpotent. Both S and N coincide, on finite-dimensional subspaces of M, with polynomials in X. Now let L be a locally finite Lie algebra of algebraic l.t. in an arbitrary vector space M whose enveloping algebra contains no nonzero nil ideals. Then L can be expressed as a direct sum of its center C and an ideal L_1 containing $[L, L]$, and possessing no nonzero solvable ideals. The l.t. in C are all semi-simple. This result can be used to prove a generalization of Lie's theorem, namely that every solvable Lie algebra of algebraic l.t. whose enveloping algebra contains no nonzero nil ideals is commutative. It is proved also that every solvable algebra of algebraic l.t. is locally finite. Finally the concept of a linearly splittable solvable algebra of algebraic l.t. is introduced, making use of the decomposition theorem for algebraic l.t. stated above for locally algebraic l.t., and some theorems are proved generalizing certain results obtained by Malcev (*Solvable Lie algebras*, Izvest. Akad. Nauk SSSR, Ser. mat. vol. 9 (1945) pp. 329–352). (Received September 16, 1954.)

7. Walter Feit: *On a conjecture of Frobenius.*

Let $\mathfrak{G}$ be a group of order $g=mq$, with $(m, q)=1$, and let $\mathfrak{M}$ be the set of all elements in $\mathfrak{G}$ whose order divides m. Frobenius conjectured that if $\mathfrak{M}$ contains exactly m elements it is a normal subgroup of $\mathfrak{G}$. He was able to prove this under the additional assumptions that $\mathfrak{G}$ contains a subgroup $\mathfrak{Q}$ of order q which is disjoint from all its conjugates and is its own normalizer. The theorem is proved by the author without the assumption that $\mathfrak{Q}$ is its own normalizer. The proof is very different from the proof Frobenius gave of his theorem; the basic tool is R. Brauer's theorem on the characterization of characters. With the help of this a representation of $\mathfrak{G}$ is constructed whose kernel is $\mathfrak{M}$. If one assumes that $\mathfrak{Q}$ is solvable, then it is possible to prove the theorem without the use of character theory. (Received September 15, 1954.)

8. N. J. Fine and Bertram Kostant (p): *The group of formal power series under iteration.*

The authors study the topological group, under iteration, of formal power series $a_1x+a_2x^2+\cdots$, a_n complex, $a_1\neq 0$ (component-wise convergence), principally by means of its "Lie algebra" $\mathfrak{g}$. The elements of $\mathfrak{g}$ are represented as operators $\phi(x)d/dx$ (cf. S. A. Jennings, Canadian Journal of Mathematics vol. 6 (1954)), and correspond under exponentiation to one-parameter groups. Following are some results: (1) Classification of all one-parameter groups (real or complex), and of elements on and off them. (2) Determination of centralizers for elements and one-parameter groups. (3) Determination of all ideals in $\mathfrak{g}$ and of all normal subgroups of G. (4) There exists a 1-1 correspondence between closed connected subgroups of G and closed subalgebras of $\mathfrak{g}$. (5) Complete solution of conjugacy problem, generalizing the Schroeder equation (cf. C. L. Siegel, Ann. of Math. vol. 43 (1942)). (6) For any representation of G into a Lie group (there exist sufficiently many), closed connected subgroups go into analytic groups. (7) Every connected Lie group imbedded in G is isomorphic to a subgroup of the complex $ax+b$ group. (8) Determination of all mth roots of any element. (9) Determination of all homomorphisms of G into itself; the automorphism group is generated by inner automorphisms and complex conjugation. (10) If $X=\phi(x)d/dx$, ϕ with positive radius of convergence, then exp tX also has positive radius of convergence. (Received September 15, 1954.)

9. I. N. Herstein: *A theorem concerning three fields.*

The following theorem is proved: let $L\supset K\supset F$ be three fields where the containing relations are all proper; suppose further that for every x in L there exists a nontrivial polynomial $f_x(t)$ in the variable t with coefficients coming from F and which depend on x, such that $f_x(x)$ is in K. Then either L is purely inseparable over K, or L, and so K, is algebraic over F. This is closely related to recent results of Herstein, Kaplansky, Krasner and Nagata, Nakayama, and Tuzuku. (Received August 26, 1954.)

10*t*. Bjarni Jónsson: *Distributive sublattices of a modular lattice.*

If X is a nonempty subset of a modular lattice A, then each of the following conditions implies that X generates a distributive sublattice of A: 1. $(\sum_{i=1}^m x_i)\prod_{j=1}^n y_j = \sum_{i=1}^m (x_i\prod_{j=1}^n y_j)$ whenever m and n are positive integers and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in X$. 2. X consists of four elements x, y, z, and u such that $(y+z)u = yu+zu$, $(z+u)x=zx+ux$, $(u+x)y=uy+xy$, $(x+y)z=xz+yz$, $(x+y+z)u=xu+yu+zu$, $xyz+u=(x+u)(y+u)(z+u)$. 3. X is the union of nonempty chains $X_1, X_2, \cdots, X_p$ such that, for every $x_1\in X_1, x_2\in X_2, \cdots, x_p\in X_p$, the set $\{x_1, x_2, \cdots, x_p\}$ generates a distributive lattice. 4. $X=B\cup C$, where B and C are distributive sublattices of A such that $(b_1+b_2)c=b_1c+b_2c$ and $(c_1+c_2)b=c_1b+c_2b$ whenever $b, b_1, b_2\in B$ and $c, c_1, c_2\in C$. (Received September 9, 1954.)

11*t*. Joachim Lambek: *Initial segments of positive semigroups.*

In a recent paper [*What is an angle*? Amer. Math. Monthly vol. 61 (1954) pp. 369–378] H. Zassenhaus abstractly characterized the additive system of all reals and all integers whose absolute value does not exceed a given bound. In the present paper, necessary and sufficient conditions are obtained, under which a set M with a partial binary operation $+$ is isomorphic with an initial segment of the positive reals or positive integers. To fix matters, assume that this segment has a largest element. The conditions then are: P1. $(x+y)+z=x+(y+z)$ if either is defined. P2. $x\neq y$ if and only if $x\in M+y$ or $y\in x+M$. P3. If $X\subseteq M$ then $X+M=M$ or $=x+M$ for some $x\in M$. P1 suffices to embed M in a semigroup S; P2 is used to extend this to a simply ordered group G with positive part S; P3 shows that G is complete. (Received September 15, 1954.)

12*t*. Joachim Lambek and Leo Moser: *On integers n relatively prime to f(n).*

It is well known that, if m and n are integers chosen at random, the probability that they are relatively prime is $6/\pi^2$. This result may still hold if m and n are functionally related. Given a sequence $f(n)$ of non-negative integers, let $Q_f(x)$ be the number of $n \leqq x$ with $(n, f(n)) = 1$. G. L. Watson has recently shown that (1) $\lim_{x\to\infty} Q_f(x)/x = 6/\pi^2$, when $f(n) = [\alpha n]$ for irrational α [Canadian Journal of Mathematics vol. 5 (1953) pp. 451–455]. Let $f^*(n)$ be the number of m for which $f(m) = n$, and suppose f and f^* are nondecreasing. It is then proved that (2) $Q_f(x) = 6x/\pi^2 + O((f(x) + f^*(f(x))) \log f(x) + x/f(x))$. Thus, for example, if $f(x) = [x^{1/2}]$, then $Q_f(x) = 6x/\pi^2 + O(x^{1/2} \log x)$. Conditions on f are obtained which imply (1), and their necessity and independence is discussed. (Received July 26, 1954.)

13*t*. R. C. Lyndon: *Representation of relation algebras.* II.

The investigation begun in RRA (Lyndon, Ann. of Math. vol. 51 (1950) pp. 707–729) is extended, and an error detected by Alfred Tarski and Dana Scott is corrected. Contrary to Theorem IV of RRA, Tarski has shown that the class of representable relation algebras is definable by universal sentences. His proof provides no construction for such a set of axioms. An extension of the method of RRA yields explicitly such a set of axioms. The mistaken result of RRA is reinterpreted correctly in terms of strong representations. The chief new tool is the consideration of divisible properties, dual to local properties, and exploitation of the known fact that the Stone-Jónsson-Tarski completion $\bar{A}$ is compact under the topology defined by A. (Received July 30, 1954.)

14. Irving Reiner: *Maximal sets of involutions.*

Let U_n denote the unimodular group; a maximal size abelian subgroup of involutions in U_n is called briefly a "maximal set." Then every maximal set has 2^n elements, given by the collection $\{MDM^{-1}\}$, where D ranges over all 2^n diagonal matrices having ± 1's as diagonal elements, and where M is a certain integral matrix. Maximal sets generated by M and M_1 are conjugate in U_n if and only if there is a relation of the form $M_1 = AMB$, where $A \in U_n$ and where B permutes columns, possibly changing their signs. A canonical form for M under $M \to AMB$ is obtained; this leads to a discussion of the nonconjugate maximal sets, and permits a group-theoretic characterization of the elements $\pm [(-1) + I^{(n-1)}] \in U_n$ up to conjugacy. The basic tool used is the connection between an involution W and the matrix whose first p columns form an integral basis for the lattice vectors in $W^+ = \{x : Wx = x\}$, and whose remaining q columns form such a basis for $W^- = \{x : Wx = -x\}$. (Received September 15, 1954.)

15. W. F. Reynolds: *On finite groups related to permutation groups of prime degree.*

Let $\mathfrak{G}$ be a finite group such that (1) $\mathfrak{G}$ contains an element of prime order p which commutes only with its own powers, and (2) $\mathfrak{G}$ is its own commutator group. Then the order g of $\mathfrak{G}$ is of the form $g = pq(np+1)$, where $np+1$ is the number of p-Sylow subgroups of $\mathfrak{G}$, and q divides $p-1$. In Ann. of Math. vol. 44 (1943) pp. 57–79, R. Brauer listed all such groups for which $n < (p+3)/2$; in unpublished work he has extended this enumeration to $n \leqq p+2$. It is now shown that no further such groups are encountered for $p+2<n<2p-3$; while for $n=2p-3$, the group $LF(2, 2p-1)$ occurs whenever $2p-1$ is a prime power. (This is an abstract of the writer's doctoral dissertation, written at Harvard University under the direction of Professor Richard Brauer.) (Received September 13, 1954.)

16. R. D. Schafer: *Noncommutative Jordan algebras of characteristic* 0.

The algebras satisfying $(xa)x=x(ax)$ and $(x^2a)x=x^2(ax)$ are here called noncommutative Jordan algebras. They are identical (for characteristic $\neq 2$) with flexible Jordan-admissible algebras, and they include such well-known algebras as Jordan, alternative, quasiassociative, and—trivially—Lie algebras. For characteristic 0 they are proved trace-admissible, and it follows from Albert, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 317–322, that any semisimple algebra (that is, an algebra having maximal nilideal$=0$) is a direct sum of simple ideals which are (a) (commutative) Jordan algebras, (b) flexible algebras of degree two, or (c) quasiassociative algebras. (Received August 18, 1954.)

17. I. M. Singer: *Automorphisms of finite factors.*

Let M be a finite factor of the type constructed by Murray and von Neumann (Ann. of Math. vol. 37 (1935) pp. 192–209) when a discrete group G acts ergodically on a measure space X. The subgroup of the automorphism group of M leaving the maximal abelian algebra of multiplications setwise invariant is analyzed. This subgroup decomposes into a semidirect product of two groups K and S. The normal subgroup K can be described as follows: The action of G on X induces an action of G on the group of measurable functions on X of absolute value 1. K is the group of 1-cocycles relative to this action, and K modulo the inner automorphisms in K is the first cohomology group. (See Eilenberg and MacLane, Ann. of Math. vol. 48 (1947) pp. 51–78.) The group S can be identified with a subgroup of the group of all measure preserving transformations on X and contains the normalizer of G. The above analysis exhibits many outer automorphisms. (Received September 15, 1954.)

18*t*. Michio Suzuki: *On finite groups with cyclic Sylow subgroups for all odd primes.*

Let G be a finite group with cyclic Sylow subgroups for all odd primes. The following results are proved. If G is simple, and if 2-Sylow subgroups are of dihedral type, then G is isomorphic with $LF(2, p)$. If G is nonsolvable and if 2-Sylow subgroups are generalized quaternion groups, then G contains a normal subgroup of index 1 or 2, which is a direct product of $SL(2, p)$ and a solvable group. Together with the results of Zassenhaus (Abh. Math. Sem. Hamburgischen Univ. (1936)), we have a complete classification of finite groups, all of whose abelian subgroups are cyclic. The theory of group characters is used in the proof. (Received September 16, 1954.)

19. George Whaples: *The generality of local class field theory.*

Call a field an r.r.f. if it satisfies the axioms for a residue class field of a field over which local class field theory holds—i.e. if (1) it has no inseparable extension and (2) it has for each n exactly one extension of degree n (in its algebraic closure). It is proved: 1. Every field of prime characteristic is contained in an r.r.f. 2. Every absolutely algebraic field of prime characteristic is maximal absolutely algebraic subfield of some r.r.f. (Received September 27, 1954.)

ANALYSIS

20. Joseph Andrushkiw: *Probability of roots of a real quadratic equation whose absolute value is less than a positive constant c.*

A real quadratic equation $xz^2+wz+y=0$ has real roots in the interval $(-c, c)$, if and only if the coefficients x, y, w are coordinates of a point which lies outside or in the surface of the cone $w^2-4xy=0$ and between the tangent planes $c^2x+cw+y=0$ and $c^2x-cw+y=0$. Denoting by v the volume of the portion of the cube $-k \leqq x \leqq k$, $-k \leqq y \leqq k$, $-k \leqq w \leqq k$ which lies outside the cone and between the above tangent planes, the author defines the probability that the quadratic equation has real roots in $(-c, c)$ by $P_r = v/8k^3$. Thus one obtains: 1. $0 \leqq c \leqq 1/2$, $P_r = c^3/9$; 2. $1/2 \leqq c \leqq 1$, $P_r = (11+6 \ln 2c)/144 + (c-c^2)/4$; 3. $1 \leqq c \leqq (5^{1/2}+1)/2$, $P_r = (11+6 \ln 2c)/144 - (c^6 - 3c^5 + 3c^2 - 1)/12c^3$; 4. $(5^{1/2}+1)/2 \leqq c \leqq 2$, $P_r = (71+6 \ln 2c)/144 - (c+1)/4c^2$; 5. $2 \leqq c < \infty$, $P_r = (41+6 \ln 2)/72 - (9c^2+c)/18c^3$. The roots are imaginary and their absolute value is less than c if and only if the coefficients x, y, w are coordinates of a point which lies inside the cone between the plane $y = c^2x$ and the cone. Defining the probability in similar way one finds: 1. $0 \leqq c \leqq 1/2$, $P_i = (2c^3)/9$; 2. $1/2 \leqq c \leqq 1$, $P_i = (36c^2 - 5 - 6 \ln 2c)/144$; 3. $1 \leqq c \leqq 2$, $P_i = (67 - 6 \ln 2c)/144 - 14c^2$; 4. $2 \leqq c < \infty$, $P_i = (31 - 6 \ln 2)/72 - 2/9c^2$. The probability P of roots of a real quadratic equation whose absolute value is less than c is given by $P = P_r + P_i$. (Received August 31, 1954.)

21. D. G. Austin: *On* $\lim_{h_n \to 0} (f(x+h_n) - f(x))/h_n$ *for approximately derivable functions.*

A result of Auerbach (Fund. Math. vol. 11 (1928) p. 197) is generalized to apply to difference quotients. It is shown that if $f(x)$ has a finite approximate derivative almost everywhere on a measurable set K, then there is a sequence of positive numbers δ_n with $\lim_n \delta_n = 0$ such that for any sequence h_n with $|h_n| \leqq \delta_n$ one has $\lim_n (f(x+h_n) - f(x))/h_n$ equal to the approximate derivative of $f(x)$ for almost all x in K. (Received September 16, 1954.)

22. W. G. Bade: *On completeness of Boolean algebras of projections in Banach spaces.*

Let X be a real or complex Banach space and $\mathfrak{B}$ be a Boolean algebra (B.A.) of projections in X. $\mathfrak{B}$ will be called *complete* (*σ-complete*) if for each family (sequence) $\{E_\alpha\} \subseteq \mathfrak{B}$, $X = \mathfrak{M} \oplus \mathfrak{N}$ where $\mathfrak{M} = \text{clm}\{E_\alpha X\}$, $\mathfrak{N} = \bigcap_\alpha (I - E_\alpha)X$, and the resulting projecting is in $\mathfrak{B}$. $\mathfrak{B}$ may be complete (σ-complete) as an abstract B.A. but not in the stronger sense above. Results: (1) if $\mathfrak{B}$ is σ-complete as an abstract B.A., then $\mathfrak{B}$ is bounded, i.e. $|E| \leqq M$, $E \in \mathfrak{B}$. (2) If $\mathfrak{B}$ is σ-complete it may be imbedded in a smallest complete B.A., the closure of $\mathfrak{B}$ in the strong operator topology. (3) The algebra generated in the uniform topology by a complete B.A. of projections is weakly closed. (Received September 14, 1954.)

23. R. G. Bartle (p), Nelson Dunford, and J. T. Schwartz: *Weak compactness and vector measures.*

Let $ca(\Sigma)$ be the Banach space of all finite countably additive (scalar) measures on a σ-field Σ, with the total variation as norm. (1) The following are equivalent for bounded subsets of $ca(\Sigma)$: (a) conditional weak compactness, (b) uniform countable additivity, (c) uniform absolute continuity with respect to some positive measure.

A function $\mu:\Sigma\rightarrow X$, a Banach space, is a *vector measure* if $x^*\mu\in ca(\Sigma)$, $x^*\in X^*$. (2) A vector measure maps into a weakly compact set. (3) A Lebesgue theory of integration of (unbounded) scalar functions with respect to a vector measure is developed. (4) If S is compact Hausdorff, an operator $T:C(S)\rightarrow X$ is weakly compact iff there is a vector measure μ on the Borel sets of S to X such that $T_f=\int f(s)\mu(ds)$. (5) T is compact iff μ maps into a compact set. Simple measure-theoretic arguments are used to derive some recent theorems of A. Grothendieck. (Received August 30, 1954.)

24. F. E. Browder: *Asymptotic distribution of eigenvalues and eigenfunctions for the general self-adjoint elliptic boundary value problem.*

Let K be a self-adjoint linear elliptic differential operator of order $2m$ on a bounded open set D of Euclidean n-space with an m-times differentiably smooth boundary. Let $a(x, \xi)$ be the characteristic form of K, $\rho(x)=\int_{a(x,\xi)<1}d\xi$. It is established for a general class of boundary value problems previously defined by the author which includes the classical Neumann and Robin problems for second-order equations, that if $\{\phi_i\}$ is a complete orthonormal sequence of eigenfunctions of $(-1)^mK$ arranged in nondecreasing order of the corresponding (real) eigenvalues $\{\lambda_i\}$, then $N(t)=\sum_{\lambda_i<1}1 \sim(2\pi)^{-n}(\int_D\rho(x)dx)t^{n/2m}$, while $N^{-1}\sum_{i=1}^{N}\phi_i(x)\phi_i(y)\rightarrow\delta_{x,y}\rho(x)\ (\int_D\rho(x)dx)^{-1}$ as $N\rightarrow\infty$ for x, y in D. Analogous results are obtained for strongly elliptic systems of differential equations. (Received September 16, 1954.)

25. D. S. Carter: *On a class of minimum-maximum problems.*

Let Y be the class of absolutely continuous real n-component functions $y(t)$ defined on an interval I, which assume given values at both end points of I, and such that a given n-component differential expression $F(t, y, \dot{y})$ has a finite maximum-norm $\langle F\rangle=\sup_{i=1}^{n}(\text{ess. }\sup_{t\in I}|F_i|)$. The problem concerns the existence and uniqueness of a $y_0\in Y$ which minimizes $\langle F\rangle$. The chief result is for the linear case $F\equiv A(\dot{y}+By+c)$. Under rather weak summability conditions on the matrices A, B and the vector c, there exists a minimizing function $y_0(t)$, which can be found by maximizing an auxiliary function of $n-1$ variables. The components of $F_0=F(t, y_0, \dot{y}_0)$ are uniquely determined, and equal in absolute value to $\langle F_0\rangle$, on certain subsets J_{i0} of I, at least one of which is of positive measure. Similar results are obtained in case the class Y is restricted by requiring the components of F to vanish a.e. on given subsets of I. These facts shed some light on the solution of the nonlinear case. (Received September 16, 1954.)

26t. Sarvadaman Chowla: *Concerning the zeros of zeta functions.*

Solving a problem suggested by Apostol, the author obtains an infinite series for the smallest complex zero of Riemann's zeta function $\zeta(s)$. (Received September 17, 1954.)

27t. R. B. Davis: *The fourth boundary value problem for third order composite equations.*

Previous work on the fourth boundary value problem has depended upon classical results concerning the behavior of the derivatives of Green's functions near the boundary of the region. The validity of these results is open to question. In the present paper, the fourth boundary value problem is established for a special third order composite equation, for the case where the region is a circle. In this special case it is possible to avoid the use of the doubtful results just mentioned. Moreover, for such a special case one can give reasonably strong uniqueness theorems. The method used is a modification of a method due to Lichtenstein (Math. Ann. vol. 67 (1909) pp. 559–575). (Received September 15, 1954.)

28. R. E. Gomory: *Trajectories tending to a critical point in 3-space.*

Consider the system $dv/dt=F(v)$ where F is a real vector. Suppose a solution $v(t)$ tends to a critical point P as $t\rightarrow\infty$. If v is a 2-vector it is well known that as $v(t)$ approaches P it either spirals infinitely often around P, or else tends to P asymptotically tangent to some line through P. However, if v is a 3-vector the behavior can be much more varied. This behavior is investigated by projecting the motion $v(t)$ onto a unit sphere D around P and studying the limit sets of the projected motion. These limit sets are not arbitrary point sets, but turn out to be the union of trajectories of a certain autonomous 2-dimensional system defined on D. By analysing this 2-dimensional system a series of results is obtained about $v(t)$. For example it is proved that the limit set of the projected motion either contains critical points of the 2-dimensional system, or else is a closed curve. This means for $v(t)$ itself that either it returns infinitely often arbitrarily close to certain special lines through P, or else it tends to P spiraling asymptotically to a cone. (Received September 10, 1954.)

29*t*. R. E. Gomory and Felix Haas: *Trajectories near limit cycles in three-space.*

An autonomous system of three differential equations with analytic right-hand side is studied. The manner in which a solution curve can tend toward a limit cycle of such a system is investigated. It is shown that under fairly general hypothesis either of two cases holds: (1) The solution curve approaches a limiting direction in each surface of section to the limit cycle. (2) All directions in every surface of section are approached by the solution curve arbitrarily late in its path and hence arbitrarily close to the limit cycle. The main difficulty which had to be overcome was that the shadow of the solution curve on a torus surrounding the limit cycle corresponded to a non-unique trajectory, which prevented a straight application of the Denjoy theory. However, another torus was constructed by expanding the limit cycle on which there were unique trajectories. It is shown that the limit set of the shadow on the first torus is the same as the limit set which the solution curve approaches on the second torus. This second limit set is then studied. (Received September 8, 1954.)

30. P. E. Guenther: *Weak solutions of difference equations.* Preliminary report.

The functional equations $f(x+h)-f(x)=g(x)$ and $f(qx)-f(x)=g(x)$ and their respective, generally divergent, formal series solutions are studied in the real domain by the methods of distribution theory. Conditions on the given function $g(x)$ are determined for each type of difference operation in order that the respective equation have weak solutions in the sense of Mikusinski. Properties of the resulting "weak sums" are compared with those of more classical solutions of summation problems. (Received September 15, 1954.)

31*t*. Felix Haas: *Morse type inequalities for the limit sets of an ordinary differential equation.*

A two-dimensional closed manifold and a vector field V defining a differential equation are studied. Certain inequalities for the trajectories of the resulting differential equation are proved. It is shown that if the limit set consists of singular points, limit cycles, and limiting configurations (closed curves with a preferred side made up of singular points and trajectories) only, then the first Betti number of the manifold minus one is smaller than or equal to the number of hyperbolic sectors plus the number of stable limit cycles plus the number of unstable limit cycles plus two times the number of stable-unstable limit cycles plus the minimum of (the number of stable limiting configurations, the number of unstable limiting configurations) minus the number of sector points. If, in addition, directed closed curves made up of singular points and trajectories are ruled out, then there exists at least one stable singular point or center point or stable limit cycle or stable-unstable limit cycle. (The important point here is that stable sectors are not sufficient.) The proof involves a deformation process and repeated use of the Meyer-Vietoris sequence. (Received September 8, 1954.)

32. H. G. Haefeli: *Runge's theorem and the correspondence between linear quaternion functionals and their indicatrices.*

A right analytic quaternion function w, regular in the open, connected set M, is the limit of a sequence of rational functions w_i. This sequence converges uniformly on each closed set in M. If $\complement M$ is connected and not finite, w is the limit of a sequence of polynomials. The local left analytic function (u, N) is orthogonal to the local right analytic function (w, M) with $M \supset \complement N$, if $\int_C w(x)dXu(x)=0$, where C is a 3-dimensional closed surface in $M \cap N$, separating $\complement N$ from $\complement M$. If (u, N) is orthogonal to every (w, M) of a linear region (A) [i.e. $M \supset A$, $A=\bar{A}$], then $u(x)=0$ for all $x \subset \complement A \subset N$. Hence a linear functional $\mathcal{L}$ has a unique indicatrix $u(x)$ such that $\mathcal{L}[w] = \int_C w(x)dXu(x)$. The function (u, N) is orthogonal to the linear subregion $(A)_C$ of all complex analytic functions of two complex variables regular in A [$\infty \subset \complement A$], if the coefficients of the Laurant expansion around infinity satisfy the conditions $\sum_{k=0}^{n-n_1} (i_1)^k c_{n_1, n-n_1-k, k}=0$ for all n and $n_1<n$. Two linear functionals $\mathcal{L}_1$ and $\mathcal{L}_2$ with the indicatrices u_1 and u_2, which map $(A)_C$ onto the complex plane, generate the same complex functional if u_1-u_2 is orthogonal to each function in $(A)_C$. (Received September 17, 1954.)

33. A. O. Huber: *On an inequality of Fejér and Riesz.*

Let $u(z)$ be defined on the closed unit circle $|z| \leqq 1$ as a difference of two subharmonic functions $u(z)=u_1(z)-u_2(z)$. Suppose that $\mu_2(|z|<1)=\alpha<1$, μ_2 denoting the measure associated with u_2. Then the inequality $\int_{-\pi}^{+\pi} \exp\{u(e^{it})\}dt \geqq 2\cos(\pi\alpha/2) \cdot \int_{-1}^{+1} \exp\{u(\rho e^{i\theta})\}d\rho$ holds for any θ $(0 \leqq \theta < 2\pi)$. Equality takes place if and only if $0<\alpha<1$ and $u=\log|F^1(z)F(z))^{-\alpha}(1-F^2(z))^{\alpha-1}|+\text{const.}$, where F is an arbitrary conformal transformation of $|z|<1$ onto itself which maps the diameter $\arg z=\theta$, $\theta+\pi$ on the real axis. This generalizes a result of L. Fejér and F. Riesz (Math. Zeit. vol. 11 (1921) pp. 305–314) and its extension by E. F. Beckenbach (J. London Math. Soc. vol. 13 (1938) pp. 82–86). The theorem can be extended to more general regions by conformal mapping and it implies a new inequality for meromorphic functions. This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 13, 1954.)

34. R. C. James: *Projections in the space (m).*

Any separable Banach space can be embedded in the space (m) of all bounded sequences. R. S. Phillips has shown that the space (c) is not complemented in (m), while A. Sobczyk used this to show that a separable Banach space is not complemented if it has a subspace isomorphic with (c_0). These results are extended to include all separable, nonreflexive Banach spaces with an unconditionally convergent basis. It is also shown that no projection of (m) onto a separable subspace can be of norm 1. (Received September 15, 1954.)

35t. Bjarni Jónsson: *A unique decomposition theorem for binary relations.*

For the notion of a sum of binary relations (ordered systems) over a binary relation, see e.g. Day, *Arithmetic of ordered systems*, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 1–43. Consider a nonempty class Φ of reflexive binary relations such that $R \in \Phi$ is implied by each of the following conditions: 1. R is isomorphic to a subrelation of a member of Φ. 2. R is a sum whose index system and terms belong to Φ. 3. Every finite subrelation of R belongs to Φ. A binary relation R is called Φ indecomposable if $R \cong \sum_{i,s} R_i$, $s \in \Phi$, implies that all but one of the R_i's are null. The author shows that every binary relation has, up to isomorphism, a unique representation as a sum over a relation in Φ of Φ indecomposable relations. This includes known results on unordered sums, linearly ordered sums (A. Tarski and the author) and sums over squares (C. C. Chang and A. Tarski). Every decomposition $R = \sum_{i,s} R_i$ induces a partitioning of the field A of R, hence an equivalence relation over A. These equivalence relations form a complete lattice $\mathcal{A}$. Decompositions with $s \in \Phi$ correspond to a principal dual ideal of $\mathcal{A}$. (Received September 9, 1954.)

36t. R. V. Kadison: *Isomorphisms of factors of type* II_∞.

It is shown that there are *-automorphisms of factors of type II_∞ with II_1 commutants which are not unitarily induced. In fact, the following theorem is proved. The group of unitarily induced automorphisms of a factor of type II_∞ with a II_1 commutant is a normal subgroup of the group of all *-automorphisms of the factor, and the quotient group is (canonically) isomorphic to the fundamental group of the factor (cf. F. J. Murray and J. von Neumann, *On rings of operators*. IV, Ann. of Math. vol. 44 (1943) pp. 716–808, see, especially, pp. 740–742). A linking operator is defined for a *-isomorphism between two rings of type II_∞ with II_1 commutants, and it is shown that a *-isomorphism is implemented by a unitary transformation if and only if its linking operator is the identity. This completes our information concerning the question of when an isomorphism between rings of operators arises spatially (cf. pp. 742–756 of paper noted above and Chapter III of E. L. Griffin, *Contributions to the theory of rings of operators*, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 471–504). (Received September 17, 1954.)

37t. Bertram Kostant: *Representations of a Lie algebra on Hilbert space.* Preliminary report.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and U_x a (strongly continuous) unitary representation of G on the Hilbert space $\mathcal{H}$. By Stone's theorem, if $\mathcal{D} = [\xi \in \mathcal{H} \mid \lim_{t \to 0} (U_t - 1)/t)\xi = \eta$ exists$]$ and U_t is such a representation of the additive real numbers R then $\mathcal{D}$ is dense and the operator A on $\mathcal{D}$ where $A(\xi) = \eta$ is skew-adjoint. When $U_t = U_{\exp tX}$ where $X \in \mathfrak{g}$, let $\mathcal{D}_X = A$ and $\pi(X) = A$. Define $\mathcal{D}^1 = \bigcap_{X \in \mathfrak{g}} \mathcal{D}_X$. $\mathcal{D}^1$ is dense. If $E_1, \cdots, E_m$ is a basis of $\mathfrak{g}$, $\mathcal{D}^1 = \bigcap_{i=1}^m \mathcal{D}_{E_i}$, $\pi(\sum_{i=1}^m \alpha_i E_i) = \sum_{i=1}^m \alpha_i \pi(E_i)$ on $\mathcal{D}^1$. The author shows $\xi \in \mathcal{D}^1$ if and only if the function $(U_x\xi, \xi)$

is of class C^2 on G. Moreover if $g(t)$ is any curve (class C^2) on G defining X at e for $t=0$, then $((U_{g(t)}-1)/t)\xi\to\pi(x)\xi$ and this characterizes $\mathcal{D}^1$. Let $\mathcal{D}^\infty=[\xi\in\mathcal{H}\mid\pi(X_1)\pi(X_2)\cdots\pi(X_k)\xi$ is defined for any finite sequence $X_1, X_2, \cdots, X_k\in\mathfrak{g}]$. $\xi\in\mathcal{D}^\infty$ if and only if $(U_x\xi, \xi)$ is of class C^∞. $\xi\in\mathcal{D}^\infty$ is well behaved (sense of Harish-Chandra) if $\sum_{n=0}^\infty(\pi(X)^n/n!)\xi$ converges for X in a neighborhood of zero. (Received September 15, 1954.)

38t. G. R. MacLane: *On the Peano curves associated with some conformal maps.*

Salem and Zygmund (Duke Math. J. vol. 12) proved that there exists a function $f(z)$, holomorphic in $|z|<1$, continuous in $|z|\leqq 1$, such that the curve $w=f(e^{it})$, $0\leqq t<2\pi$, fills some square. Further examples have been given by Piranian, Titus, and Young (Michigan Math. J. vol. 1) and Schaeffer (Duke Math. J. vol. 21). All the given proofs are arithmetical. The present paper proves a similar theorem by constructing a suitable Riemann surface onto which $|z|<1$ is mapped by $w=f(z)$, thus defining $f(z)$. In this construction it is easy to see precisely what the Peano curve obtained is. (Received August 6, 1954.)

39t. Dorothy Maharam: *On kernel representation of linear operators.*

This paper continues the author's study of "F'-integrals" (Trans. Amer. Math. Soc. vol. 75 (1953) pp. 154–184), that is (roughly) of linear, countably additive order-preserving mappings of one function space F (satisfying the countable chain condition) in another. An F'-integral ψ has a "kernel representation" in terms of another, ϕ, if $\psi(f)=\phi(kf)$ identically, the "kernel" k being a fixed element of F. If ϕ is "full-valued" (loc. cit.) it is shown that such a k exists if and only if, for each positive f, the support of $\psi(f)$ is contained in that of $\phi(f)$. Necessary and sufficient conditions are also found for the existence of a kernel representation with the kernel drawn from a larger function space. As an application, it is shown that an "F'-operator of bounded variation" (the difference between two F'-integrals) can always be represented in the form $\psi(f)=g'$ where $g'(x)=\int k(x, y)f(x, y)dy$, after a suitable isomorphic imbedding of F in a product space; conversely, any operator so represented is of bounded variation. More generally, countably many such operators can be represented in this form simultaneously. (Received September 2, 1954.)

40t. M. D. Marcus: *Asymptotic behavior of linear systems.*

Consider the vector-matrix differential equation (1) $\dot{x}=(A+B(t))x$, A a constant n-square matrix, $B(t)=U(t)+iV(t)$ a continuous complex valued n-square matrix defined on $[0, \infty)$, and x a complex n-vector. Let $A^*=$conjugate transpose of A, $U'=$transpose of U. Results: (i) Assume that maximum eigenvalue $(A+A^*)=\omega$; there exists T such that $t\geqq T$ implies either (a) $(1/t)\int_0^t \max_i U_{ii}(s)ds\leqq-\omega/2$, $\lim_{t\to\infty}\int_0^t(|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds<\infty$, $i\neq j$, or (b) $(1/t)\int_0^t(\max_i U_{ii}(s)+(1/2)\sum_{i\neq j}|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds\leqq-\omega/2$, then in either case (a) or (b) every solution of (1) is uniformly bounded as $t\to\infty$. If in (a) or (b) the integrals are bounded strictly below $-\omega/2$, then every solution of (1) converges to 0 as $t\to\infty$. (ii) Assume minimum eigenvalue $(A+A^*)=\delta$ and $\lim\inf_{t\to\infty}(1/t)\cdot\int_0^t(\min_i U_{ii}(s)-(1/2)\sum_{i\neq j}|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds>-\delta/2$, then every solution of (1) diverges to ∞ as $t\to\infty$. Special results for A triangular and for a single nth order homogeneous linear differential equation reducible to type (1) are given as well. (Received September 17, 1954.)

41t. Emanuel Parzen: *A generalization of Bernstein's probabilistic proof of Weierstrass's theorem.* Preliminary Report.

The E_r denote an r-dimensional Euclidean space, and let T be an index set. For each t in T, let $X_n(t)$ for $n=1, 2, \cdots$ be independent r-dimensional random variables with common distribution function $G(\mathbf{x}, t)$, whose means $m(t)=\int_{E_r} x dG(\mathbf{x}, t)$ satisfy the condition (1) $\int_{\|\mathbf{x}\|>M} \|\mathbf{x}\| dG(\mathbf{x}, t) \to 0$ as $M\to\infty$ uniformly in t. Let $G_n(\mathbf{x}, t)$ be the distribution function of $(1/n)\sum_{i=1}^{n} X_i(t)$. Let $g(\mathbf{x})$ be a bounded Borel function on E_r such that (2) $|g(\mathbf{x})-g(m(t))|\to 0$ uniformly in t as $\|\mathbf{x}-m(t)\|\to 0$. Using recent results of the author on the uniform convergence of families of sequences of distribution functions (University of California Publications in Statistics vol. 2 (1954) pp. 23–53), it is easy to show that $\int_{E_r} g(\mathbf{x}) dG_n(\mathbf{x}, t) \to g(m(t))$ uniformly in t. Bernstein's proof of Weierstrass's theorem on the uniform approximation of continuous functions by polynomials follows from the foregoing theorem by taking T to be the closed unit cube in E_r, $g(\mathbf{x})$ to be a function continuous on T, and $X(t)$ to be the Cartesian product of r independent binomial random variables $X^j(t_j)$ with parameter t_j. Other generalizations and applications to analysis are also indicated. (Received September 15, 1954.)

42. H. O. Pollak (p), G. H. Wannier, and D. J. Dickinson: *On a class of polynomials orthogonal over a denumerable set.*

If a set of polynomials $\Phi_n^\nu(x)$ $(\nu, n=0, 1, 2, \cdots, \Phi_0^\nu(x)=1, \Phi_1^\nu(x)=x)$ satisfies $\Phi_{n+1}^\nu(x)=\Phi_n^\nu(x)-\lambda_{n+\nu}\Phi_{n-1}^\nu(x)$ with $\lambda_n<B^\nu/(n^{1+\epsilon})$ for positive constants B and ϵ, then as $n\to\infty$, $x^n\Phi_n^\nu(x^{-1})\to E^\nu(x)$, where $E^\nu(x)$ is an entire function whose zeros are real and simple. Also, $\sum \Phi_n^\nu(x)\Phi_m^\nu(x)E^{\nu+1}(x^{-1})[dxE^\nu(x^{-1})/dx]^{-1}=\delta_{m,n}\prod_{i=1}^{n}\lambda_{i+\nu}$ where the summation is taken over those x that are zeros of $E^\nu(x^{-1})$. (Received September 15, 1954.)

43. R. A. Raimi: *Compact transformations and the k-topology in Hilbert space.*

Let H be a Hilbert space (complete but not necessarily separable), S the unit ball $\{x \mid \|x\| \leqq 1\}$, and K any compact subset of H. Then there exists a compact (completely continuous) transformation $T: H\to H$ such that $T(S)\supset K$. It follows from this lemma that the following two topologies for H are identical: (a) The *k-topology* of Arens (R. Arens, *Duality in linear spaces*, Duke Math. J. vol. 14 (1947) pp. 747–794), in which a basis set of neighborhoods of the identity is formed of the sets $\{x\in H \mid |(x, y)| \leqq 1$ for all $y\in K$, K any norm-compact set$\}$, and (b) The topology generated by a basis of sets of the form $\{x\in H \mid \|T_i(x)\| \leqq 1, T_i$ any compact transformation, $i=1, 2, \cdots, n\}$. (Received August 6, 1954.)

44. Jenny E. Rosenthal: *A new form for the solution of Laplace's equation in cylindrical coordinates.*

An extension of complex potential methods to the solution of potential problems with axial symmetry was reported previously by the author (Physical Review vol. 95 (1954) p. 633). The general class of functions obtained by this extension process has the form $W=\int f(K)[(z-iK)^2+r^2]^{-1/2}dK$. This is now studied in some detail, and its relationship is shown to standard forms of the solution of Laplace's equation. The suitability of this expression for W as a solution of boundary value problems is discussed, in particular its use in cases where standard methods fail, such as in problems with discontinuous boundaries. (Received September 15, 1954.)

45. L. A. Rubel: *Entire functions and Ostrowski sequences.*

A set S of positive integers is "an Ostrowski sequence of block ratio at least λ," where $\lambda > 1$, provided that there exist sequences $\{m_k\}$ and $\{n_k\}$, $k = 0, 1, 2, \cdots$, of positive integers satisfying (i) $\lambda m_k < n_k < m_{k+1}$, and (ii) if $m_k < n \leqq n_k$, then n is contained in S. It is shown that if an entire function of exponential type t less than π is not identically zero, yet vanishes over an Ostrowski sequence of block ratio at least λ, then t must exceed a certain function $t(\lambda)$ of λ. Two proofs of this theorem are offered, one utilizing the methods of harmonic measure, and the other using Carleman's Theorem, each of which yields an estimate of the restricting function $t(\lambda)$. Entire functions of type less than π, which vanish over specified Ostrowski sequences, are constructed, their rates of growth are estimated, and are compared with the theorems restricting them. (Received August 31, 1954.)

46. Walter Rudin: *On a problem of Collingwood and Cartwright.*

If f is meromorphic in the interior U of the unit circle, the cluster set of f, in the large, is the set $C(f)$ which consists of all w (including ∞) for which there is a sequence $\{z_n\}$ such that $z_n \in U$, $|z_n| \to 1$, $f(z_n) \to w$ as $n \to \infty$. The set of all values which f assumes infinitely many times in U is denoted by $R(f)$. Every $C(f)$ is a continuum; Collingwood and Cartwright (Acta Math. vol. 87 (1952) p. 123) have raised the question whether the converse is true. The question is answered negatively by constructing a plane continuum which is not a $C(f)$; the problem of finding necessary and sufficient conditions under which a continuum is a $C(f)$ remains open. The sets $R(f)$ are characterized as follows: E is an $R(f)$ if and only if E is the intersection of a countable sequence of *connected* open subsets of the Riemann sphere, each contained in the preceding one. In particular, every closed set is an $R(f)$, but there are open sets which are not. (Received August 25, 1954.)

47. V. L. Shapiro: *Cantor-type uniqueness of multiple trigonometric integrals.* II.

Let $c(u)$ be a complex-valued function integrable on every bounded domain in n-dimensional euclidean space E_n ($n \geqq 2$). Suppose that the integral $\int_{E_n} e^{i(x,u)} c(u)du$ is spherically summable $(C, 1)$ to zero almost everywhere and that the $(C, 1)$ spherical mean of this integral of rank R, $\sigma_R^{(1)}(x)$, is such that $\limsup_{R\to\infty} |\sigma_R^{(1)}(x)| < \infty$ in $E_n - Z$ where Z is a closed set of vanishing capacity. Suppose, further, that either (a) $c(u)(|u|^2+1)^{-1}$ is in L_1 on E_n or (b) $c(u)(|u|^2+1)^{-1}$ is in L_2 on E_n and $\int_{E_n - D_n(0,1)} e^{i(x,u)} c(u)|u|^{-2}du$ converges spherically to a continuous function in E_n where $D_n(0, 1)$ is the n-dimensional unit sphere. It is then shown in this paper that $c(u)$ vanishes almost everywhere. This theorem extends results previously obtained by the author. The essential innovation, which enables one to obtain these stronger results, is a theorem on the differentiability of Fourier transforms. This theorem is in turn based on a result of Bochner concerning the Fourier transform of spherical harmonics. Some special theorems concerning uniqueness are also obtained for the plane. (Received September 1, 1954.)

48. Domina E. Spencer: *On the classification of Bôcher equations.*

The ordinary second-order differential equations, occurring when the Laplace and Helmholtz equations are solved by separation of variables, can be classified in terms of Bôcher equations (M. Bôcher, *Über die Reihenentwickelungen der Potentialtheorie*, Leipzig, 1894). These equations are of interest in applied mathematics. The paper suggests a new method of classification of Bôcher equations and points out difficulties in the classification employed by Ince (E. L. Ince, *Ordinary differential equations*, London, 1927, Chap. XX). The effect of transformations of variables and the problem of making the specification unique are handled. (Received September 8, 1954.)

49t. W. R. Wasow: *On the convergence of an approximation method of M. J. Lighthill.*

In the Philosophical Magazine (7) vol. 40 (1949), Lighthill introduced a perturbation technique that is useful for the solution of certain differential equations near their singular points. In the present paper the mathematical validity of this method is investigated for the differential equation $(x+\alpha u)du/dx+q(x)u=r(x)$. If $q(0)>0$ and a few simple conditions are satisfied, then the solution for which $u(1)=b$ is shown to permit a parametric representation of the form $u=\sum_{s=0}^{m} u_s(\xi)\alpha^s$, $x=\xi+\sum_{s=1}^{\infty} x_s(\xi)\alpha^s$, where m is arbitrary, $u_0(1)=b$, $x_s(1)=u_s(1)=0$, $s>0$. The series for x converges if $|\alpha| \leqq \gamma_m \xi q^{(0)}$, $0\leqq\xi\leqq 1$, where γ_m is a constant. The $x_s(\xi)$, $u_s(\xi)$ are found by following essentially Lighthill's scheme. If $q(0)<0$ a variant of Lighthill's procedure using the system $\xi dx/d\xi=x+\alpha u$, $\xi du/d\xi=r(x)-q(x)u$ has certain advantages. If $r(0)=0$, which according to Lighthill entails no loss of generality, it leads to a series solution $x=\xi+\sum_{s=1}^{\infty} x_s(\xi)\alpha^s$, $u=\sum_{s=0}^{\infty} u_s(\xi)\alpha^s$ convergent for $|\alpha\xi^\rho|\leqq\gamma$, where $\rho=\min(-q(0), 1)$. The proofs use the method of dominating series. (Received August 17, 1954.)

50t. W. R. Wasow: *Singular perturbations of boundary value problems for nonlinear differential equations of the second order.*

Let the differential equation $\epsilon y''=F_1(x, y, \epsilon)y'+F_2(x, y, \epsilon)$ satisfy the following conditions. (A) The "reduced differential equation" $F_1(x, y, 0)y'+F_2(x, y, 0)=0$ possesses a solution $y=u(x)$ for which $u(\beta)=l_2$ and $F_1(x, u(x), 0)<0$ in $\alpha\leqq x\leqq\beta$. (B) The functions $F_j(x, y, \epsilon)$, $j=1, 2$, are regular analytic with respect to y and ϵ and of class C^2 with respect to x in a region R of the (x, y, ϵ)-space that contains in its interior all points $y=u(x)$, $\alpha\leqq x\leqq\beta$, $\epsilon=0$. It is shown that if ϵ and $u(\alpha)-l_1$ are sufficiently small, the full differential equation possesses a solution satisfying the boundary conditions $y(\alpha)=l_1$, $y(\beta)=l_2$ and representable by a convergent series of the form $y(x, \epsilon)=u(x)+\sum_{r=1}^{\infty} u_r(x, \epsilon)\epsilon^r$ whose coefficients satisfy certain linear differential equations and can approximately, for small ϵ, be calculated by quadratures. If the differential equation is analytic in x also, this series solution can be rearranged into the form $\sum_{m=0}^{\infty} \exp \{m\int_\alpha^x F_1(t, u/t), 0)dt/\epsilon\}\, a_m(x, l_1, \epsilon)$ where the $a_m(x, l_1, \epsilon)$ are analytic in x and l_1 and possess asymptotic expansion in powers of ϵ. These results differ from related ones by Coddington and Levinson [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 73–81] and by N. I. Brish [Doklady Akad. Nauk SSSR vol. 95 (1952) pp. 429–432] in that they permit an effective construction of the solution. (Received June 28, 1954.)

51. John Wermer: *Maximal subalgebras and Riemann surfaces.*

Let B be a commutative Banach algebra and M a proper closed subalgebra of B. M is called *maximal* if for every closed subalgebra M' with $M\subseteq M'$, either $M'=M$ or $M'=B$. Let $\mathfrak{F}$ be a Riemann surface, $\mathfrak{M}$ a region on $\mathfrak{F}$ bounded by a simple closed analytic curve γ such that $\mathfrak{M}+\gamma$ is compact. Let C be the Banach algebra of all continuous complex-valued functions on γ. Let $\mathfrak{A}$ be the subalgebra of C consisting of those f in C which may be continued into $\mathfrak{M}$ to be analytic in $\mathfrak{M}$. It is easy to see that $\mathfrak{A}$ is a proper closed subalgebra of C, containing a unit and separating points on γ. *Theorem*: $\mathfrak{A}$ is a maximal subalgebra of C. This generalizes a result of the author (Bull. Amer. Math. Soc. Abstract 59-3-306) on boundary values of functions analytic in the unit disk. (Received September 16, 1954.)

52. Albert Wilansky (p) and Karl Zeller: *Summation of bounded divergent sequences, topological methods.*

A is a summability matrix; $c_A = \{x \mid Ax \text{ is convergent}\}$; c is the space of convergent sequences. Assume $c_A \supset c$, hence $m_A \supset m$, where $m_A = \{x \mid Ax \text{ is bounded}\}$, m is the space of bounded sequences. c_A, m_A are given (F) metrics, e.g. if A is a triangle, $\|x\| = \sup_n \mid \sum_k a_{nk}x_k \mid$. *Theorem* I. *The following conditions are equivalent*: i. *c is closed in c_A*, ii. *m is closed in m_A*, iii. *A sums no bounded divergent sequences.* Tropper's result (Mathematical Reviews vol. 15, p. 118) follows. Close to known results is *Theorem* II. *If* $\lim_n \inf (|a_{nn}| - \sum(k \neq n)|a_{nk}|) > 0$, *$A$ sums no bounded divergent sequence, and c_A is the smallest linear space containing c and $\{x \mid Ax = 0\}$. ($c_A = c$ if A is reversible.)* Results of Mazur and of Petersen of the type $c_A = c \oplus$ one sequence follows. *Theorem* III. *There exists a regular row-finite A with c_A the smallest space including c and n pre-assigned sequences (independent over m). There exists a regular row-finite matrix which sums c and a pressaigned sequence of sequences (independent over m), and no bounded divergent sequence.* This improves results of Tolba (Mathematical Reviews vol. 14, p. 369) and Darevsky. (Received September 14, 1954.)

53. Mishael Zedek: *Fejér's theorem on the zeros of extremal polynomials generalized.* Preliminary report.

J. L. Walsh suggested the study of generalized Chebyshev polynomials useful for approximation of given polynomials by polynomials of lower degree and to be defined as follows: Let E be a compact point set in the complex z-plane. Of all polynomials of degree n and of the form $p_n^s(z) = z^n + A_1 z^{n-1} + \cdots + A_s z^{n-s} + a_{s+1} z^{n-s-1} + \cdots + a_n$ having the first $s+1$ coefficients $1, A_1, \cdots, A_s$ given, the author denotes by $T_n^s(z)$ one solving the extremal problem $\{\text{Max } |p_n^s(z)|, z \text{ on } E\} = \text{minimum}$. $T_n^0(z)$ are known as Chebyshev polynomials. Fejér proved (Math. Ann. vol 85 (1922) pp. 41–48) that the zeros of $T_n^0(z)$ lie in the convex hull of E. The main result for the zeros of $T_n^s(z)$ states that the sum of the angles subtended by E at any group of $s+1$ of them is $\geqq \pi$. (Received September 15, 1954.)

Applied Mathematics

54*t*. Peter Henrici: *An eigenvalue problem in the theory of viscous flow.*

L. Collatz and H. Goertler (Z. Angew. Math. Physik vol. 5 (1954) pp. 95–110) reduce the study of the stationary, slightly rotational, axisymmetric flow of a viscous fluid through a straight circular tube to the following eigenvalue problem for a function $G(s)$ proportional to the angular velocity component: (1) $s^2G'' + sG' - G = -16\beta^2 s^2(1-s^2)G$, $0 \leqq s \leqq 1$; (2) $G(0) = G(1) = 0$. They determine the first few eigenvalues and eigenfunctions by numerical methods and give for the large eigenvalues the approximation $\beta_n \sim n$. The author shows that (1) can be solved in terms of the confluent hypergeometric function. This leads to the equation ${}_1F_1(1-\beta; 2; 4\beta) = 0$ for the eigenvalues and, by application of Watson's Lemma to a suitable contour integral, to asymptotic formulas for the large eigenvalues and the corresponding eigenfunctions. In particular, $\beta_n = m + Am^{-4/3} + Bm^{-8/3} + O(m^{-10/3})$, where $m = n + 1/6$ and A and B are certain constants. Additional nonasymptotic information about the eigenfunctions is obtained from a theorem of Pólya and Sonine (see Szegö, *Orthogonal polynomials*, p. 161). Numerical results are given. The work was supported by the Office of Naval Research. (Received September 13, 1954.)

55. Peter Henrici: *Application of two methods of numerical analysis to the computation of the reflected radiation of a point source.*

The integral $f(x, y) = \int_0^\infty dr \int_0^{2\pi} (1 + x^2 + r^2 - 2xr \cos \phi)^{-3/2} (r^2 + y^2)^{-2} r d\phi$, which arises in the theory of radiation and which was to be computed numerically for many values of x and y, is reduced to a finite simple integral involving the hypergeometric function $F(3/4, 5/4, 1, Z)$ where (*) $0 \leqq Z < 1$. In the first part of the paper the effect of Aitken's δ^2-method (Proc. Roy. Soc. Edinburgh vol. 57 (1936–37) pp. 269–304) upon the summation of the hypergeometric series is studied. It is shown that if (*) holds the application of the method reduces the truncation error after n terms by a factor k, where $1 \leqq 8n^3(1-Z)^2 k \leqq 2^{-3/2}\pi$. The number of terms necessary for a given accuracy is reduced similarly, and the "ratio of practical convergence" (remainder over first neglected term) is not affected adversely by the method. In the second part the error induced by numerical quadrature of the simple integral is estimated by a method due to P. Davis and P. Rabinowitz (see Journal of Rational Mechanics and Analysis vol. 2 (1953) pp. 303–313, and NBS Report No. 3270). (Received September 13, 1954.)

56t. Peter Henrici: *On helical springs of finite thickness.*

The problem of determining the stress distribution in a statically loaded, closely coiled helical spring, whose (circular) cross-section is not necessarily small in comparison with the diameter of the helix, has been solved approximately by an iterative method by Goehner (Ing. Arch. vol. 1 (1930) pp. 619–644) and exactly in terms of a series of appropriate Legendre functions by Freiberger (Aust. Journ. of Scient. Res. A vol. 2 (1949) pp. 354–375). The author shows that the exact values of the stress concentration factor as well as some other significant quantities of the problem can be developed in terms of a certain parameter (namely the reciprocal of the spring index). Thus Goehner's approximations are recovered and further approximations of the same type are obtained. The analytical work involves an application of Clausen's identity (J. Reine Angew. Math. vol. 3 (1828) pp. 89–95) to the computation of the Taylor expansion of the square of a Legendre function. The work was supported by the Office of Naval Research. (Received September 13, 1954.)

57t. R. S. Ledley: *A general method for introducing constraints on the generating propositions of a propositional calculus of symbolic logic.*

Digital computational methods for the propositional calculus of symbolic logic, or the analogous free Boolean algebra, were given by the author (National Bureau of Standards Report 3363). These methods associate a binary designation number of 2^n digits to each of n generating propositions (elements) E_i: where if $(E_i)_j$ represents the jth position digit of the designation number for E_i, then $(\overline{E}_i)_j = 0, 1$ when $(E_i)_j = 1, 0$; $(E_i \cap E_j)_k = (E_i)_k \odot (E_j)_k$ where $0 \odot 0 = 1 \odot 0 = 0 \odot 1 = 0$, $1 \odot 1 = 1$; $(E_i \cup E_j)_k = (E_i)_k \oplus (E_j)_k$ where $1 \oplus 1 = 1 \oplus 0 = 0 \oplus 1 = 1$, $0 \oplus 0 = 0$. Also, representing E_i by E_i^1, $\overline{E}_i$ by E_i^0, consider the 2^n possible intersections $\bigcap_{i=1}^{i=n} E_i^c$ where $c = 0$ or 1, for the

2^n possible dispositions of the n superscripts c in the formula; then $(\bigcap E_i^c)_h=1$ for only one integer h, $1 \leqq h \leqq 2^n$, and 0 otherwise, where h differs for each of the 2^n possible intersections. In this Boolean algebra the generating propositions are logically independent. However in practical problems one often desires to introduce logical constraints or dependencies between the generators. If $f_i(E_1, E_2, \cdots)$, $i=(1, \cdots, m)$, represent the Boolean functional constrains applied, then the designation numbers of the E_i are reduced to include only those positions k for which $(\bigcap_{i=1}^{i=m} f_i)_k=1$. The computational methods for a many component propositional logic is an application of this general constraint method. All the results and computational methods for the free Boolean algebra are equally valid in any constraint case. (Received August 26, 1954.)

58t. R. S. Ledley: *A many-component propositional, two-valued logic.*

The familiar propositional calculus of symbolic logic involves propositions with two components in a two-valued logical system. However, practical logical problems in science, industry, and government frequently deal with many-component propositions, where the ith component, Q_i, of a proposition Q having r_k components is defined by $\overline{Q}_i=\bigcup_{j=1}^{j=r_k} Q_j$ and $Q_i \cap Q_j=0$ both for $j \neq i$. The author has previously presented a digitalization and systematization of the free Boolean algebra with 2^{2^n} elements (see Journal of the Operations Research Society of America vol. 2 (1954)), which can be interpreted in terms of the two-component propositional, two-valued logic generated by n propositions. This same digitalization has now been extended to include non-free Boolean algebras with 2^n elements. These algebras can be interpreted as representing many-component propositional, two-valued logics, where for a system of m propositions with $r_1, r_2, \cdots, r_m$ components each, $n=\prod_{i=1}^{i=m} r_i$. All results and computational methods of the free Boolean case are equally valid in this non-free case. The number of units in the binary designation number of length n for a single component of the kth proposition is n/r_k. The digitalized techniques enable these propositions to be manipulated with the same facility as ordinary algebraic type functions, the methods providing easy implementation by an electronic "logic machine." (Received August 26, 1954.)

59. R. S. Ledley: *Digitalization and systematization of some aspects of the functional calculus of symbolic logic.*

Aspects of the functional calculus are digitalized and systematized to achieve straightforward computational methods enabling solution of those practical problems arising in industry, science, and government, which need the functional as well as the propositional logical calculus. The theory is based on elementary universally valid formulas ($i, j, \cdots$ representing universal, $m, n, \cdots$ existential quantifiers) such as: $x_iy_jP(x, y) \equiv y_jx_iP(x, y)$; $x_my_iP(x, y) \supset y_ix_mP(x, y)$; $x_iP(x) \supset x_mP(x)$. Formulas are treated as expressions in a modified prenex normal form; the prefix rspresented by a_{mi}, where $a_{mi}=1$ for x_i in the scope of y_m, $a_{mi}=0$ otherwise. The weight of the mth row is $\sum_i a_{mi}$; that of the ith column is $\sum_m \bar{a}_{mi}$. The predicates themselves are handled by the author's digitalized method for the propositional calculus. With this notation, systematic methods are formulated for: generating all implications of any number of expressions deducible by elementary formulas; testing any number of expressions for contradictions, redundancy and tautology; manipulating and analyzing functional expressions; obtaining the solution to expression equations, etc. Both the restricted and extended functional calculus are treated. This digitalization is amenable to easy mechanization by an electronic "logic machine." The notation is extended to include additional class types of quantifiers such as 'there exist at least n,' etc. where for r types, the array is $a_{i_1,\ldots,i_r}$. (Received August 24, 1954.)

60*t*. Eric Reissner: *On transverse vibrations of thin shallow elastic shells.*

It is shown that in the range of frequencies governing *transverse* vibrations of shallow shells the differential equations of the problem may be simplified by omitting *longitudinal inertia* terms. The practical importance of this simplification is illustrated by explicitly determining the frequencies of free transverse vibrations of simply-supported shells with second-degree middle surface equation and rectangular base-plane projection of the boundary. (Received August 19, 1954.)

61*t*. Eric Reissner: *On axi-symmetrical vibrations of shallow spherical shells.*

The paper determines explicitly the frequencies of free, axi-symmetrical, transverse vibrations of shallow spherical shell segments with clamped or free edges. The calculations further indicate the practical advantages of being able to neglect longitudinal inertia in problems of this type (see preceding abstract). A comparison is made with an earlier study of the problem of the shell segment with clamped edge, carried out without neglecting longitudinal inertia (J. Appl. Phys. vol. 17 (1946) pp. 1038–1042. (Received August 19, 1954.)

GEOMETRY

62*t*. Louis Auslander and L. F. Markus: *Holonomy of flat affinely connected manifolds.*

Let M be a differentiable (C^∞) n-manifold with a differentiable affine connection Γ which is flat (torsion and curvature tensors are zero). Then M admits a covering by local coordinates such that on each intersection of coordinates the Jacobian matrix is constant and thus linear phenomena, i.e. linear differential equations with constant coefficients, are meaningful relative to this covering. Conversely, if M admits a covering by local coordinates with constant Jacobian matrices, then M admits a differentiable flat affine connection. For differentiable n-manifolds M with flat affine connection Γ there exists a homomorphism from the fundamental group $\Pi_1(M)$ onto the holonomy group $H(M; \Gamma)$. In the principal bundle $B(M)$ the set of bases joined by "horizontal" curves to a given base defines a holonomy covering space $\widehat{M}$ for M. With the raised connection $\widehat{\Gamma}$, $H(\widehat{M}; \widehat{\Gamma})=0$ and the covering transformation group of $\widehat{M}$ over M is $H(M; \Gamma)$. In a certain sense, $\widehat{M}$ is the (minimal) universal covering space of M with regard to holonomy. Finally, suppose $H(M; \Gamma)=0$ and Γ is complete. Then M is a Riemannian manifold with Christoffel connection Γ and M is isometric (differentiably) with E^n modulo a discrete vector subspace. (Received September 16, 1954.)

63. C. C. Hsiung: *On the total curvature of a simple closed curve in a Riemannian manifold.*

Let C be a simple closed curve in an n-dimensional Riemannian manifold V_n ($n \geqq 2$) which is twice differentiably imbedded in a Euclidean space E_m of m ($>n$) dimensions, and let κ_g be the first curvature of C relative to V_n. Furthermore, let $N_{n+1}|,\ldots,N_m|$ be any $m-n$ mutually orthogonal vectors normal to V_n at a point P,

$\kappa_{\nu|1}, \cdots, \kappa_{\nu|n}$ the principal curvatures of V_n associated with any normal vector $N_{\nu|}$ ($\nu = n+1, \cdots, m$), and M the first mean curvature of V_n at P defined by $M^2 = \sum_{\nu=n+1}^{m} (\sum_{h=1}^{n} \kappa_{\nu|h})^2$. If $\kappa_{\nu|1}, \cdots, \kappa_{\nu|n}$ for each ν have the same sign at each point of C, then it is shown that (*) $\int_C |\kappa_g| ds \geqq 2\pi - \int_C |M| ds$, where s is the arc length of C. For a Euclidean V_n, the inequality (*) was obtained by W. Fenchel for $n=3$, and independently by B. Segre, H. Rutishauser and H. Samuelson, and K. Borsuk for a general n. (Received September 15, 1954.)

64. Valdemars Punga: *On the determination of affine connection in metric spaces. Application to Cartan-Finsler space.*

The usual way of determining the affine connection in a metric space is to set the covariant differential of a metric tensor equal to zero (i.e. $\delta g_{\alpha\beta} = 0$) and to solve the corresponding equation for $\Gamma^{\alpha}_{\beta\gamma}$. The disadvantages of this method are: (a) we should know the formula for the covariant differential of a metric tensor, (b) in more general metric spaces the metric is not always defined by the tensor $g_{\alpha\beta}$. Here is a method for the determination of an affine connection for any metric: equate the length of the vector at the initial point (or at an initial line element) to the length of the same vector when parallelly displaced to the neighbouring point (or neighbouring line element) and solve the resulting equation for the affine connection using infinitesimals up to the first order. If the space is defined so that these two lengths are not equal (as in Weyl space), then use the method of linear displacement of length (see V. Punga, *The principle of linear displacement of length and Weyl's geometry*, in this Bulletin). This method will be demonstrated for Cartan-Finsler space of line elements. The length of a vector $v^{\alpha}(x, x')$ in Cartan-Finsler space is given by $v = (g_{ij}(x, x')v^i v^j)^{1/2}$. The vector at the line element $(x^{\alpha} + dx^{\alpha}, x'^{\alpha} + dx'^{\alpha})$ parallel to the vector v^{α} at $(x^{\alpha}, x'^{\alpha})$ is the vector $v^{\alpha} + Dv^{\alpha} = v^{\alpha} - \Gamma^{*\alpha}_{\beta\gamma} v^{\gamma} dx^{\beta} - C^{\alpha}_{\beta\gamma} v^{\gamma} \delta x'^{\beta}$, $\delta x'^{\beta} = dx'^{\beta} + \Gamma^{\beta}_{\sigma\rho} x'^{\rho} dx^{\sigma}$ (the proof that $v^{\alpha} + Dv^{\alpha}$ is a vector is similar to that of V. Punga, *On parallel displacement of a vector*, in this Bulletin). According to the author's method he sets up $g_{\alpha\beta}(x, x')v^{\alpha}v^{\beta} = g_{\alpha\beta}(x+dx, x'+dx')(v^{\alpha} + Dv^{\alpha})(v^{\beta} + Dv^{\beta}) = (g_{\alpha\beta}(x,x') + (\partial g_{\alpha\beta}/\partial x_{\gamma})dx^{\gamma} + (\partial g_{\alpha\beta}/\partial x'^{\gamma})dx'^{\gamma})(v^{\alpha}v^{\beta} + g_{\alpha\beta}v^{\beta}Dv^{\alpha} + g_{\alpha\beta}v^{\alpha}Dv^{\beta})$, or $[\partial g_{\alpha\beta}/\partial x^{\gamma} - g_{\sigma\beta}\Gamma^{*\sigma}_{\gamma\alpha} - g_{\alpha\sigma}\Gamma^{*\sigma}_{\gamma\beta} - (\partial g_{\alpha\beta}/\partial x'^{\sigma})\Gamma^{*\sigma}_{\gamma\rho}x'^{\rho}]dx^{\gamma} + [(\partial g_{\alpha\beta}/\partial x'^{\gamma}) - g_{\sigma\beta}C^{\sigma}_{\gamma\alpha} - g_{\alpha\sigma}C^{\sigma}_{\gamma\beta}] \cdot \delta x'^{\gamma} = 0$. Since dx^{γ} and $\delta x'^{\gamma}$ are independent it follows that the expressions in brackets are zero. Solving them for $\Gamma^{*\alpha}_{\rho\gamma}$ and $C^{\alpha}_{\rho\gamma}$ one obtains Cartan's formulas (E. Cartan, *Les espaces de Finsler* II). (Received September 13, 1954.)

65t. Valdemars Punga: *On parallel displacement of a vector.*

The parallel displacement of a vector v^{α} from the point x^{α} to $x^{\alpha} + dx^{\alpha}$ is defined by the affine connection $\Gamma^{\alpha}_{\beta\gamma}(x)$ with the law of transformation from coordinate system (α, β, $\cdots$ Greek indices) to (a, b, $\cdots$ Roman indices): (1) $\Gamma^{a}_{bc} = (\partial x^{a}/\partial x^{\alpha})(\partial x^{\beta}/\partial x^{b}) \cdot (\partial x^{\gamma}/\partial x^{c})\Gamma^{\alpha}_{\beta\gamma} - (\partial x^{\beta}/\partial x^{b})(\partial x^{\gamma}/\partial x^{c})(\partial^2 x^{a}/\partial x^{\beta}\partial x^{\gamma})$, $x^{a} = x^{a}(x^{\alpha})$. The vector at the point $x^{\alpha} + dx^{\alpha}$ parallel to the vector v^{α} at the point x^{α} is the vector $v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}$. It is not obvious that $v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}$ is a vector: $v^{\gamma}(x)$ is a vector, dx^{β} is a vector, but $\Gamma^{\alpha}_{\beta\gamma}(x)$ is not a tensor and hence the product $\Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}$ seems to be not a tensor. But the vector $v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}$ is located not at the point x^{α}, but at the point $x^{\alpha} + dx^{\alpha}$ and because of this location it is a vector; to prove this will be the object of this paper. Thus, one has to prove that (2) $(\partial x^{a}/\partial x^{\alpha})_{x^{\alpha}+dx^{\alpha}} \cdot (v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}) = v^{a} - \Gamma^{a}_{bc}v^{c}dx^{b}$. Assuming that the relation $f(x^{\alpha} + dx^{\alpha}) = f(x^{\alpha}) + (\partial f/\partial x^{\sigma})(x)dx^{\sigma}$ holds for any admissible function $f(x)$, one has similarly: (3) $(\partial x^{a}/\partial x^{\alpha})_{(x^{\alpha}+dx^{\alpha})} = \partial x^{a}/\partial x^{\alpha} + (\partial^2 x^{a}/\partial x^{\alpha}\partial x^{\sigma})dx^{\sigma}$. Thus the left member of (2) under substitution (3) changes to (4) $(\partial x^{a}/\partial x^{\alpha})_{x^{\alpha}+dx^{\alpha}} \cdot (v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma}v^{\gamma}dx^{\beta}) = (\partial x^{a}/\partial x^{\alpha}$

$+(\partial^2 x^a/\partial x^\alpha \partial x^\sigma)dx^\sigma)(v^\alpha - \Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta)$. Assuming (3) the author restricts the use of infinitesimals to those of the first order only (like dx^α) and hence is justified in (4) in dropping out any term containing infinitesimals of order higher than one (like $dx^\sigma dx^\gamma$). Hence (4) reduces to (5) $(\partial x^a/\partial x^\alpha)v^\alpha + (\partial^2 x^a/\partial x^\alpha \partial x^\sigma)v^\alpha dx^\sigma - (\partial x^a/\partial x^\alpha)\Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta$. Substituting $(\partial x^a/\partial x^\alpha)v^\alpha = v^a$, $v^\alpha = (\partial x^\alpha/\partial x^a)dx^a$, $dx^\sigma = (\partial x^\sigma/\partial x^s)dx^s$ into (5) and changing correspondingly the dummy indices, (5) reduces to (6) $v^a + [(\partial^2 x^a/\partial x^\gamma \partial x^\beta)(\partial x^\gamma/\partial x^c) \cdot (\partial x^\beta/\partial x^b) - (\partial x^a/\partial x^\alpha)(\partial x^\beta/\partial x^b)(\partial x^\gamma/\partial x^c)\Gamma^\alpha_{\beta\gamma}]v^c dx^b$. The expression in brackets according to (1) is Γ^a_{bc}, so that (6) reduces to $v^a + \Gamma^a_{bc}v^c dx^b$. Q.E.D. (Received September 13, 1954.)

66t. Valdemars Punga: *The principle of linear displacement of length and Weyl's geometry.*

Given a metric space ($g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$) with a symmetric affine connection $\Gamma^\alpha_{\beta\gamma}(x) = \Gamma^\alpha_{\gamma\beta}(x)$. The vector at $x^\alpha + dx^\alpha$ parallel to the vector v^α at x^α is the vector $v^\alpha + Dv^\alpha = v^\alpha - \Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta$ (see V. Punga, *On parallel displacement of a vector*, in this Bulletin). The principle of parallel displacement of vectors (and tensors of any rank) is the leading principle in modern differential geometry. At the same time the principle of parallel displacement of the length of a vector in metric spaces, taken independently from vector displacement, is not so widely appreciated. The expression "parallel displacement of length" would be replaced by "linear displacement of length." In this paper a reasonable formula for displacement of length will be obtained and it will be shown that if two independent principles of length and vector displacement are taken simultaneously, Weyl geometry results. Assume that displacing a length v_0 from a point x^α_0 to the point x^α the length v is obtained, where $v = f(x_0, x) \cdot v_0$ or, in differential form: $Dv^\alpha = v_0(\partial f/\partial x^\alpha)_{x=x_0}dx^\alpha$. Setting up $(\partial f(x, x_0)/\partial x^\alpha)_{x=x_0} = -\phi_\alpha(x_0)/2$, for any point $x = x_0$ and any vector $v = v_0$, one obtains $Dv = -(v/2)\cdot\phi_\alpha(x)dx^\alpha$. Hence $Dv^2 = 2vDv = -v^2\phi_\alpha(x)dx^\alpha$. On the other hand $Dv^2 = g_{\alpha\beta}(x+dx)(v^\alpha + Dv^\alpha)(v^\beta + Dv^\beta) - g_{\alpha\beta}(x)v^\alpha v^\beta = (g_{\alpha\beta} + dg_{\alpha\beta})(v^\alpha - \Gamma^\alpha_{\lambda\mu}v^\mu dx^\lambda)(v^\beta - \Gamma^\beta_{\sigma\rho}v^\rho dx^\sigma) = -v^2\phi_\alpha(x)dx^\alpha$. Solving this equation for $\Gamma^\alpha_{\beta\gamma}$ (up to infinitesimals of the first order), one obtains $\Gamma^\alpha_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + (\delta^\alpha_\beta\phi_\gamma + \delta^\alpha_\gamma\phi_\beta - g^{\sigma\alpha}g_{\beta\gamma}\phi_\sigma)$, which is the Weyl connection. (Received September 13, 1954.)

Logic and Foundations

67. Kurt Bing: *On arithmetical classes not closed under direct union.*

Horn (J. Symbolic Logic vol. 16 (1951) pp. 14–21) has shown that all classes of algebras characterizable by closed sentences of conditional type are closed under passage to direct union and that this result cannot be improved by allowing a larger class of characterizing sentences describable in terms of the quantifiers and propositional structure of their prenex normal forms. The problem whether all classes of algebras characterizable by closed sentences, but not by those of conditional type, fail to be closed under direct union is still unsolved. In this paper, building on a suggestion and work by Tarski (Proceedings of the International Congress of Mathematicians, Cambridge, 1950, vol. 1, pp. 705–720) and work by McKinsey (J. Symbolic Logic vol. 8 (1943) pp. 61–76), two sufficient conditions for nonclosure under direct union of arithmetical classes of algebras not characterizable by functions of conditional type, and a sufficient condition for closure, not known to be co-extensive with Horn's condition, are proved. (Received September 14, 1954.)

68t. A. R. Schweitzer: *A survey of differential equation theories and related subjects.*

The author first discusses means of transition from integral equations (considered in a previous paper) to differential equations and then classifies research in differential equations according to leading schools indicated by the following names. I. Gauss, Pfaff, Grassmann, Riemann, Fuchs, Weierstrass. II. Pincherle, Volterra. III. Lagrange, Cauchy, Lamé, Picard, Poincaré. IV. F. R. Moulton, G. D. Birkhoff. It is assumed that the subject of differential equations had its origin in the mathematical interpretations of natural phenomena such as astronomy and physics and that differential equations, as well as integral equations and integro-differential equations, are encompassed by functional equations. As a basis for orientation in the extensive domain of differential equations the author selects differential equations of the Fuchsian class. Subjects connecting investigations of Fuchs with researches of Picard, Poincaré, Birkhoff include the method of successive approximations (Picard), automorphic functions (Poincaré), and singular points of differential equations (Birkhoff). Subjects related in a wider sense to differential equations include transformation groups, boundary value problems, differential geometry, and topology. Reference is made to the treatises and papers of L. Schlesinger and to the collected mathematical papers of G. D. Birkhoff. (Received September 14, 1954.)

Statistics and Probability

69. Miriam Lipschutz: *On the approximation to stable distributions.*

Consider a sequence of identically distributed, independent, positive, continuous random variables X_k, with distribution function $F(x)$. Let $1-F(x)=h(x)/x^\gamma$, $0<\gamma<2$, where $\lim_{x\to\infty} h(cx)/h(x)=1$ for any positive constant c. The sum $S_n=\sum_{k=1}^n X_k$ properly normalized will tend to a stable distribution $G_\gamma(x)$ of index $(\gamma, -1)$. Let $r(m)\to\infty$ as $m\to\infty$. If for $x<r(m)^{3/\gamma}$, say, the expansion $h(mx)/h(m)=1+\sum_{k=1}^\infty a_k l_k(x)/r(m)^k$ be valid, then under some mild auxiliary conditions on the functions $l_k(x)$, the following results hold for $0<\gamma<1$: (a) $\operatorname{Sup}_{-\infty<x<\infty} |P(S_n<b_n x)-G_\gamma(x)|<A/r(b_n)$. (b) If $x\to 0$ as $n\to\infty$, but $x>[((1+\epsilon)/k_\gamma) \lg r(b_n)]^{(\gamma-1)/\gamma}$, then $P(S_n<b_n x)\sim Bx^{\gamma/2(1-\gamma)} \exp -k_\gamma x^{-\gamma/(1-\gamma)}$, where B and k_γ are constants. Similar results are obtained in the case $1<\gamma<2$, for $P(S_n-n\mu<b_n x)$ where $\mu=E(X)$, when x is finite and also when $x\to-\infty$. The method used is that of characteristic functions. The difficulties previously encountered by investigators in an attempt to use this method are overcome by noting that in convergence to a stable distribution the major contribution is due to the maximum term. Hence the author has first estimated the condition c.f. of S_n for a given value of the maximum. This bounds the range of integration of the other $n-1$ terms and enables the necessary expansions to be made. The probability that the maximum term exceeds this range is shown to be negligible. (Received September 16, 1954.)

Topology

70t. E. H. Brown, Jr.: *Finite computability of the homotopy groups of finite groups.*

Let G be an abelian group and n an integer. According to J. C. Moore (Ann. of Math. vol. 59 (1954) pp. 549–557) a space X is said to be of homotopy type (G, n) if and only if X is simply connected, has trivial homology except in dimension n, and $H_n(X)$ is isomorphic with G. The homotopy groups of such a space depend only on G and n and, therefore, are denoted by $\pi_i(G, n)$. The aim of this paper is to prove that $\pi_i(g, n)$ is finitely computable when G is finite, and from this to prove the finite computability of the stable homotopy groups of spheres. (Received September 24, 1954.)

71. J. R. Munkres: *A note on the Hurewicz theorem.*

Proofs of the Hurewicz theorem appeal (usually implicitly) to the homotopy addition theorem, an "obvious" proposition whose proof entails considerable complications of a technical nature (See Hu, *The homotopy addition theorem*, Ann. of Math. vol. 58, p. 108). The situation becomes somewhat simpler when one uses (as has become common in recent years) the cubic rather than the simplicial singular homology. Eilenberg's proof of the Hurewicz theorem (Ann. of Math. vol. 45, pp. 439–443) is readily adapted to this situation, and the appeals to the homotopy addition theorem may be replaced by the following special case of it, which is proved in this paper: Let u be a map of the unit $n+1$ cube, I_{n+1}, into X which maps the $n-1$ skeleton into $x \in X$. Let d be the homology boundary operator; then du may be written $\sum_1^m \epsilon_i v_i$, where $\epsilon_i = \pm 1$ and v_i is a map of $(I_n, \dot{I}_n) \to (X, x)$, so that it defines an element $[v_i]$ of $\pi_n(X, x)$. Then $[v_1]^{\epsilon_1} \circ \cdots \circ [v_m]^{\epsilon_m} = 0$, where $\circ$ is the group operation in π_n. (Received August 2, 1954.)

L. W. COHEN,
Associate Secretary