THE FEBRUARY MEETING IN NEW YORK

The five hundred eleventh meeting of the American Mathematical Society was held at Columbia University in New York City on Saturday, February 26, 1955. The meeting was attended by about 220 persons, including 191 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor Harish-Chandra of Columbia University delivered an address entitled *Representations of semisimple Lie groups* at a general session presided over by Professor Claude Chevalley. Sessions for contributed papers were held in the morning and afternoon, presided over by Professor Everett Pitcher, Dr. R. L. Sternberg, and Professor D. J. Struik.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Mr. G. A. Baker, Jr., was introduced by Professor H. A. Arnold, and Mr. Shôshichi Kobayashi by Professor C. B. Allendoerfer.

ALGEBRA AND THEORY OF NUMBERS

383. S. S. Abhyankar: *Splitting of valuations in extensions of local domains.* II.

Let R be a local domain of dimension bigger than one with quotient field K and maximal ideal M. Let K^* be a finite separable extension of K. In a previous joint note (Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955)), Zariski and the author have proved that: if R is regular and has the same characteristic as its residue field, then there exist infinitely many real discrete valuations of K having center M in R which split in K^*. In the present note the following generalization is proved: If either (a) R admits a necleus (and hence in particular if R is a geometrical local ring) or if (b) R is regular and has the same characteristic as its residue field, then there exist infinitely many real discrete valuations v of K having center M in R such that v has $[K^*:K]$ distinct extensions to K^*. Case (b) is proved by applying Zariski's recent results on normal sequences of local rings (Rend. di Mat. Roma (5) vol. 13 (1954) pp. 1–38) to the quadratic sequence along a valuation of K with center M in R which splits in K^* in order to obtain a quadratic transform of R which splits in K^*, and by passing to a corresponding splitting field. Case (a) is proved by passing to completions and applying case (b). This theorem gives a new proof of a recent result of Nagata for local domains admitting a necleus. (Received January 4, 1955.)

384t. H. W. Becker: *Lebesgue quadrilaterals with elements in common.*

There are obvious interpretations of (A) $x^2+y^2+z^2=t^2$ as quadrilaterals, parallelopipeds, or quadrirectangular tetrahedra; hence, of such with common els. LQ with

a leg in common (generalizing Hero $\triangle$) are (B) $y = Y = 2(mn - pq)(MN - PQ)$, $z = 2(mp + nq)(MN - PQ)$; t, $x = (m^2 \pm n^2 \pm p^2 + q^2)(MN - PQ)$; T, $X = (mn - pq)$ $(M^2 \pm N^2 \pm P^2 + Q^2)$; $Z = 2(mn - pq)(MP + NQ)$. It is easy then to commonize another el., especially $z = Z$. (A) with (C) $2y^2 + z^2 = s^2$ has solution t, $x = 5a^2 \pm b^2$; s, y, $z = 6ab$, $4ab$, $2ab$, or more generally, t, $x = (a^2 + c^2)(5a^2 - 4ac + c^2) \pm b^2$; s, y, $z = 2b(3a^2 - 2ac + c^2)$, $4ab(a - c)$, $2b(a^2 + 2ac - c^2)$. General solution of LQ with hypotenuse t in common is based on Euler's forms (1) Dickson's *History* II, p. 277. LQ with hyp. and one leg in common are furnished by permutation or sign change of Lebesgue's parameters, ibid. p. 265. Ibid. p. 270 erratum: "We may express 1521 as a 3⑤ in 7 ways" should be 8 ways. If (D) $X^2 + Y^2 + Z^2 = T^2$, $T^2 + U^2 + V^2 = W^2$ (generalizing Eels quads.) then X, Y, Z, T, U, V, $W = x_1x_2$, x_1y_2, x_1z_2, x_1t_2, y_1t_2, z_1t_2, t_1t_2, the subscripted letters satisfying (A). (D) transforms: X, Y, Z, T, U, V, $W \to TX$, UX, VX, WX, WY, WZ, WT; T^2, TU, TV, TW, UW, VW, W^2; X^2, XY, XZ, XT, YT, ZT, T^2. Solutions of $t^2 \mp (y^2 + z^2) = x^2$, w^2 are t, y, z, x, $w = 41$, 24, 12, 31, 49; 97, 96, 12, 7, 137; 97, 84, 48, 7, 137; 185, 132, 120, 49, 267; 185, 168, 60, 49, 267 (also $49^2 + 168^2 = \square$). Brahmagupta quads. generalize to octagons: X, T, Z, $Y = x_1x_2$, t_1x_2, z_1x_2, y_1x_2; X', T', Z', $Y' = X$, x_1t_2, x_1z_2, x_1y_2; X'', T'', Z'', $Y'' = y_1y_2$, t_1y_2, z_1y_2, Y'; X^*, T^*, Z^*, $Y^* = X''$, y_1t_2, y_1z_2, Y. A closed chain of 4 LQ each pair having 2 legs in common is an O'Riordan vector, ibid. pp. 502–505. (Received January 10, 1955.)

385t. H. W. Becker: *Pythagorean 4-facets*. Preliminary report.

These are Diophantine solutions of (A) $u^2 + w^2 = r^2$, $x^2 + y^2 = u^2 = u^2$, $z^2 + o^2 = w^2$, with (B) $y^2 + z^2 = v^2$. The Pyth. parameters are $\langle a, b; c, d; e, f; g, h \rangle$. By E. T. Bell, Scripta Math. vol. 12 (1946) pp. 88–89, the general solution of (A) is r, $u = k(a^2 \pm b^2)$ $(c^2 + d^2)(e^2 + f^2)$; w, $z = 2kab(c^2 + d^2)(e^2 \pm f^2)$; $x = k(a^2 - b^2)(c^2 - d^2)(e^2 + f^2)$, $y = 2kcd$ $(a^2 - b^2)(e^2 + f^2)$, $o = 4abefk(c^2 + d^2)$. Also satisfying (B), but proscribing (C) $x^2 + o^2$ $= q^2$ (a rearranged Brahmagupta quad., RBQ) there are 13 primitive numerical solutions with $r < 1000$, the smallest being the mavericks $\langle 8, 5; 3, 2; 2, 1; 2, 1 \rangle$, $\langle 13, 2;$ $3, 1; 3, 2, 10, 1 \rangle$, $\langle 13, 8; 2, 1; 3, 2; 5, 2 \rangle$. 3 of them generalize to the parametric solutions $\langle m^4 - m^2n^2 + n^4, (mn)^2; m^2, n^2; m^2 - n^2 \pm mn; m, n \rangle$, $\langle m^4 + n^4, 2(mn)^2; m, n; m^2, n^2; m, n \rangle$, $\langle (m^2 + n^2)^2, (2mn)^2; m^2 + n^2, 2mn; m, n; m, n \rangle$. There are numerous transforms to RBQ, like x, y z, o, r, u, v, $w \to xu$, yu, yw, xw, ru, u^2, yr, wu; ou, zu, zw, ow, rw, uw, rz, w^2; yu, yw, zw, zy, rv, ry, vw, rz; yu, zu, zw, yw, rv, uv, rz, wv; xy, xz, yz, y^2, uv, xv, uz, vy; yo, yz, z^2, zo, vw, yw, zv, zw; xz, xo, yo, yz, uw, xw, ou, yw: under which $\langle a, b; c, d; w, f; g, h \rangle \to \langle a, b; c, d; c \pm d; a, b \rangle$, etc. A RBQ further satisfying (D) $x^2 + y^2 + z^2 = t^2$ is the Pyth. hexahedron x, y, z, $o = XY$, Y^2, YZ, ZX, where (E) $V^2 + X^2 = U^2 + Z^2 = T^2$, $V^2 - Z^2 = U^2 - X^2 = Y^2$ is a Pyth. tet. If additionally to (A, B, D), (F) $y^2 + z^2 + o^2 = s^2$ is met, the system would be a Pyth. decahedron (all 10 facets Pyth. $\triangle$) but it is conjectured this is impossible. If instead of (B), (B') $y^2 - z^2$ $= v'^2$, such 4-facets are rare: e.g. x, y, z, o, $r = 204$, 85, 36, 48, 229; 104, 195, 48, 36, 229. (Received January 10, 1955.)

386t. J. L. Brenner: *Relations among the minors of a matrix with dominant principal diagonal.*

If the absolute value of the principal element of each row of a square matrix exceeds the sum of the absolute values of the nonprincipal elements of that row, it is known that the matrix is invertible. This article establishes general inequalities among the minors of such a matrix. Two corollaries of the main theorem are the following. First, a lemma of Ostrowski, which he describes as a remarkable inequality [Proc. Amer. Math. Soc. vol. 3 (1952) p. 26]. Second, the absolute value of the principal

minor on any set of rows exceeds the absolute value of any nonprincipal minor on the same set of rows. The main theorem is stronger and more exact. Credit is due the Office of Ordnance Research for sponsoring this work. (Received January 10, 1955.)

387t. Eckford Cohen: *Congruences in algebraic number fields involving sums of similar powers.*

Let F be a finite extension of the rational field, and suppose P to be a prime ideal of F of norm p^f. Denote by $Q_s(\rho)$ the number of solutions of the congruence $\rho \equiv \alpha_1 X_1^m + \cdots + \alpha_s X_s^m \pmod{P^\lambda}$, where ρ is arbitrary, $(m, p) = 1$, and $(\alpha_i, P) = 1$, $i = 1, \cdots, s$. A general formula for $Q_s(\rho)$, involving the generalized Jacobi sum, is obtained, and approximations to $Q_s(\rho)$ are deduced. These estimates are used to prove the following result: If $s \geq 3$, then $Q_s(\rho) > 0$ for all P of sufficiently large norm that satisfy the above restrictions. The case $s = 2$ is treated in detail and explicit formulas for $Q_2(\rho)$ are determined under certain conditions. (Received January 12, 1955.)

388t. Eckford Cohen: *An extension of Ramanujan's sum.* II. *Additive properties.*

Let $(a, b)_k$ denote the greatest common kth power divisor of a and b. The function $c_k(n, r) = \sum \exp(2\pi \, \mathrm{i}nx/r^k)$, where x ranges over the integers (mod r^k) such that $(x, r^k)_k = 1$, was introduced by the author in *Part* I (Duke Math. J. vol. 16 (1949) pp. 85–90). In the present paper the following basic orthogonality property is proved: If $d \mid r$, $e \mid r$, then $\sum c_k(a, d)c_k(b, e) = r^k c_k(n, r)$ or 0 *according as* $d = e$ or $d \neq e$, the summation being over a, b such that $n \equiv a + b \pmod{r^k}$. This property is used to obtain the number $Q_k(r)$ of solutions x_i (mod r), y_i (mod r^k), of the congruence $n \equiv a_1 x_1^k y_1 + \cdots + a_s x_s^k y_s \pmod{r^k}$, where $(a_i, r) = 1$. The result for $Q_k(r)$ is given in terms of a singular sum involving $c_k(n, r)$. An equivalent form of the above orthogonality property is also derived, and a generating function of $Q_k(r)$ is deduced. (Received February 17, 1955.)

389t. Eckford Cohen: *Some totient functions.*

Using the notation of the preceding abstract, the function $\phi_k(r)$ is defined to be the number of integers b of a complete residue system (mod r^k) such that $(b, r^k)_k = 1$. The characteristic properties of $\phi_k(r)$ are deduced arithmetically, and the function is shown to be equivalent to the totient functions of Jordan (*Traité des substitutions* (1870) pp. 95–97) and of von Sterneck (Monatshefte Math. Phys. vol. 5 (1894) pp. 255–256). The number of solutions of the semilinear congruence discussed in the preceding abstract is then obtained in terms of ϕ_k. Two proofs are given, one based on the Jordan function and the other on the von Sterneck function. Finally, a divisor relation involving ϕ_k is deduced, resolving a question proposed by Métrod (L'Intermédiaire des Math. vol. 20 (1913) pp. 148–149; Dickson, *History*, vol. I, 1918, p. 155). (Received February 17, 1955.)

390t. Eckford Cohen: *An extension of Ramanujan's sum.* III. *Connections with generalized totients.*

This paper concludes a study (*Part* II, abstract above) of the function $c_k(n, r)$, emphasizing its relation with generalized totients. In particular, it is proved that $c_k(n, r) = \phi_k(r)\mu(r/d)\phi_k^{-1}(r/d)$ where $d^k = (n, r^k)_k$, μ denotes the Möbius function, and ϕ_k is the totient of the immediately preceding abstract. This result reduces to a theorem of O. Hölder in case $k = 1$ (Prace Mat.-Fiz. vol. 43 (1936) pp. 13–23). The

orthogonality properties of *Part* II above are shown to reduce, in special cases, to certain simple arithmetical relations involving divisor sums. Some of these results are then applied in deriving a formula for the number of solutions of the linear congruence, $n \equiv a_1 x_1 + \cdots + a_s x_s \pmod{r^k}$, $(a_i, r) = 1$, in $x_i \pmod{r^k}$ such that $(x_i, r^k)_k = 1$. The formula obtained generalizes a theorem of Vandiver and Nicol in the case $k = 1$, but the methods of proof are quite different (Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 825–835). Finally, the Dirichlet series $\sum_{n=1}^{\infty} c_k(n, r) n^{-s}$, $s > 1$, is summed by the methods of the present paper. (Received February 17, 1955.)

391t. Casper Goffman: *A definition of completeness for partly ordered groups*. Preliminary report.

Let G be an ordered group. A set $S \subset G$ is locally bounded above if for every $x > e$, the identity of G, there are y, $z \in S$ such that xy and zx are upper bounds of S. G will be called complete if every set which is locally bounded above has a least upper bound. For a class of abelian groups G, including the abelian 1-groups, associated complete groups are obtained as follows: A section in G is a pair of nonempty sets (A, B) such that $x \in A$, $y \in B$ implies $x < y$. It is maximal if $C \supset A$, $D \supset B$ and (C, D) a section implies $C = A$, $D = B$. It is dedekindean if it is maximal and if for every $y > 0$, the identity of G, there is an $x \in A$ such that $x + y \in B$. With trivial identifications, the dedekindean sections form a complete ordered group $\overline{G}$. If G is complete then $\overline{G}$ is isomorphic with G. For integrally closed groups, this completion agrees with the usual one, and for totally ordered groups it agrees with one given by Cohen and Goffman (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 310–319). (Received December 3, 1954.)

392. I. N. Herstein: *Jordan homomorphisms*.

A Jordan homomorphism of one ring R into another ring R' is an additive mapping of R into R' which preserves squares. R' is said to be a prime ring if $x'R'y' = (0)$ implies $x' = 0$ or $y' = 0$. In this paper it is proved that a Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. In particular, Jordan automorphisms of simple and primitive rings of characteristic not 2 or 3 are automorphisms or anti-automorphisms. This greatly extends results of Jacobson-Rickart, Kaplansky, Hua and Ancochea. Additive mappings preserving nth powers in case R is a ring with unit element and R' is a prime ring of characteristic larger than n are shown to be homomorphisms or anti-homomorphisms multiplied by $(n-1)$st roots of unity lying in the center of R'. (Received January 10, 1955.)

393t. B. W. Jones and D. B. Marsh: *A proof of a theorem of Meyer on indefinite ternary quadratic forms*.

By free use of matrices and the modern theory of quadratic forms including the Hasse symbol, the authors obtain a relatively simple proof of a theorem of Adolf Meyer which is fundamental in the theory of quadratic forms and the determination of class number of primitive indefinite ternary genera. After demonstrating that two forms in the same genus are equivalent if they represent a binary form with certain properties, the authors determine two conditions under which a ternary form represents all binary forms of given determinant; they complete the proof by establishing the satisfaction of these conditions under the hypothesis of the theorem. (Received December 15, 1954.)

394. D. H. Lehmer: *The determination of large isolated values of Ramanujan's function.*

The function $\tau(n)$ of Ramanujan, defined as the coefficient of x^{n-1} in the power series expansion of the 24th power of Euler's product $(1-x)(1-x^2)(1-x^3) \cdots$, has been the subject of many investigations. It is a multiplicative function whose values for successive powers of a fixed prime p form a second order recurring series. Unfortunately one of the coefficients in this recurrence is $\tau(p)$, so that this leaves the knowledge of $\tau(n)$ incomplete to the extent that one does not know this function for prime values of its argument. Unlike most other well known numerical functions, $\tau(p)$ is not a polynomial function of p, although it is a polynomial with respect to various moduli, some of which are fairly large. Tables of $\tau(n)$ extend only to $n \leq 2500$. It thus becomes important to be able to obtain exact isolated values of $\tau(p)$. A formula for doing this is derived from the fact that $\tau(n)$ enters into the error term of certain divisor functions. This formula, too lengthy to be given here, has been used to compute many values of $\tau(p)$ for p as large as 13127 and 15359. One application of the formula for $p = 3967$ gives $\tau(p)/p^{11/2} = 1.9514$ which is the largest value for this ratio yet found. According to the Ramanujan hypothesis this ratio does not exceed 2 in absolute value. (Received February 17, 1955.)

395t. Irving Reiner: *Unimodular complements.*

The following theorem is due to Steinitz (Math. Ann. vol. 71 (1911) pp. 328–354): Let R be the ring of algebraic integers in an algebraic number field, and let $a_1, \cdots, a_n$ be relatively prime elements in R. Then there exists an $n \times n$ matrix with first row $(a_1 \cdots a_n)$ whose determinant is a unit in R. A simpler proof, valid for any Dedekind ring R, is given in this note. (Received January 13, 1955.)

396t. Kenneth Rogers: *Extremal lattices of convex bodies in complex space.*

In complex space C^n are defined star-bodies, lattices over complex quadratic number-fields, also admissible, extremal and critical lattices, all as natural analogues of the notions in R^n. Points P, Q are said to be nonassociated if the equation $P = \epsilon Q$ is false for all units ϵ of $k(\theta)$. As analogue of H. P. F. Swinnerton-Dyer's result (Proc. Cambridge Philos. Soc. vol. 49 (1953) p. 161), it is shown that for a bounded, closed convex body K in C^n, any extremal lattice in $k(\theta)$ has at least $n(n+1)$ nonassociated points on the boundary of K. As analogue of results of Korkine and Solotaryov (Math. Ann. vol. 11 (1877) pp. 242–292), it is shown that if M is the minimum of an extremal positive definite Hermitian form, then n^2 of the representations of M can be found which uniquely fix the form. This is used to find the least upper bound for the minimum of a positive definite ternary Hermitian form of determinant D, with variables in $k(i)$. The best possible result is $M \leq (4D)^{1/3}$, and there is just one class of forms for which the equality sign is necessary. (Received January 13, 1955.)

397t. Kenneth Rogers and H. P. F. Swinnerton-Dyer: *Lattices in complex quadratic fields.*

For lattices over complex quadratic fields, a complete analogue of Mahler's theory (Proc. Roy. Soc. London vol. A 187 (1946) pp. 151–187) for real lattices is given. In particular, the existence of a critical lattice for a star-body of finite type is proved, there being essentially new difficulties only when the class-number of the field is not 1. For the body in C^2 given by $|x| \leq 1$, $|y| \leq 1$, a finite process for determining all critical

lattices is given, thus giving a general solution of a problem Minkowski solved for the Gaussian and Eisenstein fields. In terms of forms, the result for $k(i5^{1/2})$ is: for linear forms ξ, η, of determinant 1 in variables u, v, there exist integers $(u, v) \neq (0, 0)$ of $k(i5^{1/2})$ such that max $(|\xi|, |\eta|) \leq \{2/(-3+(15)^{1/2})\}^{1/2}$. This is best possible, there being one class of critical forms. Other applications of the general theory are given. (Received January 13, 1955.)

398t. R. L. San Soucie: *A characterization of a class of rings.*

Let F be a field of characteristic two having an automorphism α of order two and an element f_0 in F such that $f_0\alpha = f_0$ and f_0 not a square in F. Define the additive endomorphism θ of F by $f\theta = f\alpha + f_0 f$, all f in F. If R is the set of all couples (f, g) with f, g in F, if equality and addition in R are defined componentwise, and if multiplication in R is defined by (*)$(f, g)(h, k) = (fh + g \cdot k\theta, fk + gh)$, with θ as defined above, then R. H. Bruck has shown that R is a not alternative, right alternative division ring of characteristic two. [See R. L. San Soucie, *Right alternative division rings of characteristic two*, to appear soon in the Proceedings of this society]. The present paper characterizes a class of right alternative division rings of characteristic two, including the rings mentioned above. It is proved that if R is a not alternative, right alternative division ring of characteristic two, if R satisfies the identity $(wx, y, z) = w(x, y, z) + (w, y, z)x + (w, x, (y, z))$, and if all commutators are in the left nucleus of R, then R is two-dimensional over some field F and multiplication in R is defined as in (*), where the θ is *some* additive endomorphism of F. (Received December 27, 1954.)

399. Ernst Snapper: *Integral closure of modules and complete linear systems.*

Let Σ/k be a finitely generated field extension, i.e., Σ and k are fields and $\Sigma = k(a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ are elements of Σ. A module L is a subgroup of the additive group of Σ, which is closed under multiplication by elements of k. One says that L has a finite number of generators if L consists of the linear combinations $c_1 s_1 + \cdots + c_m s_m$, where $s_1, \cdots, s_m$ are fixed elements of Σ and $c_1, \cdots, c_m$ vary freely in k. The product LM of two modules L and M is the smallest module which contains all the products $\lambda\mu$, where $\lambda \in L$ and $\mu \in M$. The integral closure $|L|_i$ of a module L is defined as the module which consists of those elements b of Σ for which there exists a non-zero, finitely generated module M, such that $bM \subset ML$. We prove that, if L is finitely generated, then $|L|_i$ is finitely generated. It is shown that the finite generation of $|L|_i$ is the field-theoretic equivalent of the geometric fact that every linear system of $(r-1)$-dimensional cycles of an irreducible, r-dimensional algebraic variety is contained in a complete linear system. (Received January 13, 1955.)

400. Ruth Rebekka Struik: *On associative products of groups.*

Continuing the work of O. N. Golovin [Mat. Sbornik vol. 27 (69) (1950) pp. 427–453], a doubly-infinite number of associative products of groups can be constructed in the following manner: Let $G = A_1 * A_2$ be the free product of the groups A_1 and A_2. Let (A, B) be the subgroup generated by all $a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Let R be a subgroup of G and $_kR_G$ the $(k+1)$st member of the lower central series of R, that is, $_0R_G$ is the normal subgroup generated by R and $_kR_G = (_{k-1}R, G)$. Let $A_1 \circ A_2 = (A_1 * A_2)/(_mA_{1G}, _nA_{2G}) \cdot (_nA_{1G}, _mA_{2G})$ where m, n are any two fixed non-negative integers. Then $\circ$ is associative, that is, $(A_1 \circ A_2) \circ A_3 \cong A_1 \circ (A_2 \circ A_3) \cong (A_1 * A_2 * A_3)/\prod_{i,j=1;i\neq j}^{3}(_mA_i, _nA_j)$ under the obvious mapping, for all groups A_1, A_2,

A_3. Not all commutative products are associative. It can be shown that $\bigtriangledown$ is not associative, where $A_1 \bigtriangledown A_2 = (A_1 * A_2)/((A_1, A_2), (A_1, A_2))$. The proof of associativity utilizes identities and inequalities of subgroups of free products, and in particular $_kR_F \le {}_kR_G({}_kA_{3_F} \cap {}_kG_F)$, where $F = A_1 * A_2 * A_3$. (Received January 10, 1955.)

401t. N. A. Wiegmann: *On symmetric matrices with real quaternion elements.*

It is known that a matrix A with complex elements is symmetric if and only if there exists a complex unitary matrix U such that UAU^T is a real diagonal matrix (where U^T denotes the transpose of U); here possible analogs for a matrix with real quaternion elements are considered. If U is unitary and quaternion, necessary and sufficient conditions that U^T is unitary are obtained, and certain related results are stated. It is shown that if A is quaternion and symmetric, there exists a real orthogonal matrix U such that UAU^T is quaternion and diagonal if and only if the hermitian polar matrix of A is real. Matrices of the form UDU^T are then considered where D is quaternion and diagonal and U is, in turn, complex unitary and quaternion unitary (where U^T is also unitary). (Received January 5, 1955.)

<div align="center">ANALYSIS</div>

402t. R. P. Boas: *Integrability of trigonometric series. IV.*

Let $\{\lambda_n\}$ be a sequence of bounded variation (i.e., $\sum |\Delta\lambda_n| < \infty$). Form $\mu_n = \sum' \{(-1)^{n-m} - 1\}(n-m)^{-1}\lambda_m$ where the sum omits the term $m = n$. Then if $\{\mu_n\}$ is also of bounded variation, the λ_n are Fourier coefficients of a Lebesgue integrable function. A closely related theorem is that if both $f(x)$ and $f(x)$ sgn x have absolutely convergent Fourier series over $(-\pi, \pi)$, then $f(t)/t$ is Lebesgue integrable. These theorems imply various older results of a similar character. (Received January 7, 1955.)

403t. F. E. Browder: *On the positivity and semi-boundedness of the Dirichlet forms associated with an elliptic differential operator.*

Algebraic conditions in terms of the characteristic form of a Dirichlet form, associated with an elliptic differential operator K on a domain G of Euclidean n-space, are established which are necessary and sufficient for its semi-boundedness on the space $C^m(G)$. (Received January 13, 1955.)

404t. H. S. Collins: *Completeness, full completeness, and k-spaces.*

Let E be a completely regular T_1 topological space and $C(E)$ be the real l.t.s. of continuous real-valued functions on E, with the compact-open topology. Theorem 1: E is a k-space if and only if $C(E)$ is complete and E with its k-topology (=strongest topology on E which agrees with the given topology on each compact set) is completely regular. Theorem 2: If E is either pseudo-finite or hemicompact then E with its k-topology is completely regular (by definition, E is pseudo-finite if every compact set is finite, and is hemicompact if the compact sets have a countable base). Theorem 3: For E either pseudo-finite or hemicompact the following conditions are equivalent: (1) $C(E)$ is complete; (2) E is a k-space; and (3) $C(E)$ is fully complete (in Bull. Amer. Math. Soc. Abstract 59-1-10, full completeness is termed Property A). In addition, for E pseudo-finite, each of the following is equivalent to each of (1)–(3): (4) E is discrete; (5) E is locally compact; (6) $C(E)$ is the cartesian product of E copies of the reals, with the product topology. (Received December 22, 1954.)

405t. R. B. Davis: *Asymptotic solutions of a fourth order elliptic partial differential equation.* Preliminary report.

The extension of Levinson's asymptotic study of second-order elliptic partial differential equations (Ann. of Math. vol. 51 (1950) pp. 428–445) to fourth-order equations encounters this difficulty: there is no directly analogous maximum principle available. One may circumvent this obstacle in either of two ways: by using integral norms (following a suggestion of Berg and Lax), or by restricting the equation and choosing boundary values in a suitable fashion. The present paper follows the second course, and establishes an asymptotic theorem analogous to Levinson's, for the case where the operator factors ($Lu = L_1L_2u = f$), and where boundary values are given for L_2u, as well as for u. (Received January 13, 1955.)

406. Jesse Douglas: *Comparative considerations concerning the Dirichlet principle and the Douglas principle.*

The title and contents of a recently published book entitled *Dirichlet's principle,* as well as the general tenor of some recent mathematical writings, have seemed to promote among non-connoisseurs the impression that one can solve the Plateau problem with this classic principle. All such attempts, however, extending over an 80 year period in mathematical history (1850–1930), met with failure. The fact is, of course, that the Plateau problem presents an essentially new difficulty, of higher order than the fundamental problems in complex variable and conformal mapping whose solution, following the lead of Riemann, has been based on the Dirichlet principle. An entirely fresh approach was necessary. The principle introduced by the author for the solution of Plateau's problem is: $A(g) = $ minimum, where the functional $A(g)$ is defined as the mean square of the ratio of corresponding chords of the given contour Γ and the unit circumference C (times π). The argument g is an *arbitrary* one-one continuous *map* of C onto Γ, and corresponding chords are those whose end points are g-related. In the main form of Dirichlet's principle, on the other hand, the argument ϕ in the basic functional $D(\phi)$ is a *numerical*-valued function on a given plane region with a *prescribed* value at each boundary point. The contrast between the two minimum principles is further elaborated. (Received January 12, 1955.)

407. Jim Douglas, Jr. and T. M. Gallie, Jr. (p): *Variable time steps in the solution of the heat flow equation by a difference equation.*

The numerical solution of the boundary value problem $u_{xx} = u_t$ ($0 < x < 1$, $t > 0$), $u(0, t) = u(1, t) = 0$, $u(x, 0) = g(x)$, is treated using the backwards difference equation $\Delta_x^2 w_{i,n+1} = (w_{i,n+1} - w_{in})/\Delta t_n$, where $x_i = i\Delta x$, $t_n = \Delta t_0 + \cdots + \Delta t_{n-1}$, $f_{in} = f(x_i, t_n)$, $\Delta_x^2 f_{in} = (f_{i+1,n} - 2f_{in} + f_{i-1,n})/(\Delta x)^2$. It is well known that if the ratio of Δt to $(\Delta x)^2$ is held constant, the solution of the difference equation converges to that of the differential equation with the truncation error being $O((\Delta x)^2)$. It is shown that the conclusion holds for $\Delta t_n/(\Delta x)^2 = \alpha + \beta t_n$ or $\Delta t_n/(\Delta x)^2 = \gamma \exp(\delta t_n)$, $0 < \delta < \pi^{2/2}$. This results in a great reduction in total calculation necessary to complete the numerical solution. (Received January 10, 1955.)

408. Albert Edrei: *Gap properties of entire functions of finite order, bounded on a radial path.*

Let (1) $F(z) = F(0) + \sum_{n=1}^{\infty} c_n z^{\lambda_n} [c_n \neq 0]$ be an entire function of finite order ρ, bounded for positive values of the variable z. Then, defining the lower logarithmic

density of the sequence $\{\lambda_n\}$ by $\delta = \lim \inf_{x\to\infty} (\log x)^{-1} \sum_{\lambda_n \leq x} \lambda_n^{-1}$, one has $2\rho\delta \geq 1$. Conversely, given any sequence $\{\lambda_n\}$ of positive increasing integers and of lower logarithmic density $\delta(>0)$, there exists an entire function, of the form (1), of order $(2\delta)^{-1}$, which remains bounded on the positive axis. Slight variations in the arguments used to prove the above proposition enable the author to answer completely the following question: if a real entire function, of finite order ρ, is bounded for $z > 0$, how frequent must be the changes of sign of the coefficients of its expansion $\sum a_n z^n$. (Received January 12, 1955.)

409t. H. G. Eggleston: *A property of bounded analytic functions.*

$f(z)$ is a bounded regular function in $|z| < 1$. The set of points θ, at which the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists is denoted by F. The set of θ at which the cluster set of $f(z)$ reduces to a single point is denoted by G. It is shown that $F \subset G \cup H$ where H is of first category. It follows that there exist functions $f(z)$ for which F is of first category. (First category is relative to the set $0 \leq \theta \leq 2\pi$.) (Received January 10, 1955.)

410t. A. G. Fadell: *Accessibility in Euclidean n-space.*

Let there be given a point $p = (p^1, \cdots, p^n)$ in a subset S of Euclidean n-space. The accessible set $A[p, s]$ of p is defined to consist of all points $(x^1, \cdots, x^n)$ in S which satisfy the condition (vacuous for $n = 1$) that the points $(x^1, \cdots, x^{k-1}, p^k, \cdots, p^n)$ be in S for $k = 2, \cdots, n$. The accessible set $A[p, S]$ is open, closed, or Borel according as S is open, closed, or Borel. Let $S(p^k, \cdots, p^n)$, $2 \leq k \leq n$, denote the set of points $(x^1, \cdots, x^{k-1})$ in Euclidean $(k-1)$-space such that $(x^1, \cdots, x^{k-1}, p^k, \cdots, p^n)$ is in S. Then $A[p, S]$ is L_n-measurable if S is L_n-measurable and $S(p^k, \cdots, p^n)$ is L_{k-1}-measurable for $k = 2, \cdots, n$. The main result of the paper is the following: If p is a point of an L_n-measurable set S such that $S(p^k, \cdots, p^n)$ is L_{k-1}-measurable for $k = 2, \cdots, n$, then p is a point of density for $A[p, S]$ if and only if p is a point of density for S and $(p^1, \cdots, p^{k-1})$ is a point of density for $S(p^k, \cdots, p^n)$ for $k = 2, \cdots, n$. As a corollary we have that almost every point of an L_n-measurable set is a point of density for its accessible set. The latter theorem and corollary have application in relating partial conditions to full conditions, especially in differentiability questions. (Received January 3, 1955.)

411t. A. G. Fadell: *Characterizations of the existence of a total and an approximate total differential.*

Let f be a real-valued function defined on a subset S of Euclidean n-space, and let $M = \lim \sup |f(p) - f(x)|/\|p - x\|$. Then f is termed of *bounded linear distortion* (b.l.d.) at a point p in S if $M < \infty$ as x approaches p through S. Also, f is termed of *approximately* b.l.d. at p in S if p is a point of density for a subset E of S for which $M < \infty$ as x approaches p through E. Application of the writer's results concerning accessibility (see preceding abstract) yields a proof of the n-dimensional analog of the Rademacher-Stepanoff Theorem, namely, that a real-valued L_n-measurable function f on an L_n-measurable subset S of Euclidean n-space has a (Stolz) total differential a.e. in S if and only if f is a.e. of b.l.d. in S. Application of the latter theorem along with accessibility enables the following theorem: Given a real-valued L_n-measurable function f on an L_n-measurable subset S of Euclidean n-space, the following conditions are equivalent: (1) f has an approximate total differential a.e. in S; (2) f has approximate first partial derivatives a.e. in S; (3) f is a.e. of approximately b.l.d. in S; (4) f is a.e. partially of approximately b.l.d. in S; (5) f has a total differential a.e. on a sequence of L_n-measurable sets S_k with $L_n(S - \bigcup_{k=1}^{\infty} S_k) = 0$; (6) f is Lipschitzian

on a sequence of L_n-measurable sets S_k with $L_n(S - \bigcup_{k=1}^{\infty} S_k) = 0$. The equivalence of (1) and (2) for $n = 2$ was first proved by Stepanoff. (Received January 3, 1955.)

412t. A. G. Fadell: *A characterization of the existence of a regular approximate total differential.*

A real-valued function f defined on a subset S of Euclidean n-space is said to have a regular approximate (or asymptotic) total differential at p in S if there exist finite constants A_j, $j = 1, \cdots, n$, such that $\lim |f(p) - f(x) - \sum_{j=1}^{n} A_j(p^j - x^j)| / \|p - x\|$ exists and equals zero as x approaches p through a subset E of S having the following two properties: (1) E is the intersection of S and the union of oriented n-cubes having p as center and (2) p is a point of density for E. The main result of the paper is the following: A real-valued function f defined and continuous on an oriented n-cube Q, $n \geq 2$, has a regular approximate total differential a.e. in Q if and only if f is of bounded linear distortion a.e. in Q with respect to each set of $(n-1)$ variables. Also, for an L_2-measurable function f on a planar L_2-measurable set the existence almost everywhere of a regular approximate total differential is equivalent to the existence almost everywhere of an approximate total differential. (Received January 3, 1955.)

413. Sigurdur Helgason: *A multiplier problem.*

Let $A = \{x\}$ be a semi-simple, self-adjoint Banach algebra represented by the algebra $\hat{A} = \{\hat{x}\}$ of continuous functions on the maximal ideal space $\mathfrak{M}$. Let $C^{\infty}(\mathfrak{M})$ denote the set of bounded continuous functions on $\mathfrak{M}$ and $C(\mathfrak{M})$ denote those functions in $C^{\infty}(\mathfrak{M})$ that vanish at infinity on $\mathfrak{M}$. A function $f \in C^{\infty}(\mathfrak{M})$ is called a multiplier if $\hat{x}f \in \hat{A}$ for every $\hat{x} \in \hat{A}$. Let $\tilde{A}$ denote the algebra of all multipliers. Then $C(\mathfrak{M}) \subset \tilde{A}$ implies $C^{\infty}(\mathfrak{M}) = \tilde{A}$ if at least one of the two following conditions is satisfied: (i) A is regular and every closed ideal in A is contained in a regular maximal ideal. (ii) A is reflexive. This is proved by using the derived algebra A_0 (Proc. Nat. Acad. Sci. U.S.A. vol. 40, pp. 994–995) to get a decomposition of every linear functional $F \in A'$ into homomorphisms. Thereby one can to every $F \in C^{\infty}(\mathfrak{M})$ construct an endomorphism T_f of the second dual A'' leaving A invariant and such that the effect of T_f on A corresponds to the multiplication of $\hat{A}$ by f. (Received January 13, 1955.)

414t. A. J. Lohwater: *Beurling's theorem for quasi-conformal mappings.*

Let $w = f(z)$ be a topological mapping of $|z| < 1$ with bounded dilatation coefficient. It is shown that $f(z)$ possesses unique boundary values "in angle" for all $e^{i\theta}$ on $|z| = 1$ except for a possible set of logarithmic capacity zero, and that these boundary values cannot be the same on a set of positive capacity. Extensions are made to pseudo-analytic functions which are not necessarily univalent in $|z| < 1$. These results extend certain results of Beurling [Acta Math. vol. 72 (1940) pp. 1–13] for a class of analytic functions. (Received January 6, 1955.)

415. Paul Malliavin: *Reduction of some questions of uniqueness to a Watson's problem.*

Some problems of uniqueness, gathered by S. Mandelbrojt in a same theory by the method of the adherent series, are reduced to a same problem, generalizing the Watson's problem. (Received January 11, 1955.)

416. C. B. Morrey, Jr. (p) and James Eells, Jr.: *A variational method in the theory of harmonic integrals.*

In this paper a variational method is carried through to prove the following orthogonal decomposition theorem: *Any r-form ω in L_2 on a compact orientable smooth manifold can be uniquely expressed in the form $\omega = H + \delta\alpha + d\beta$, where H is a harmonic form and α and β are forms in* $\mathfrak{p}_2$ (their components are absolutely continuous along lines and their partial derivatives are in L_2; see *Functions of several variables and absolute continuity* by Calkin and Morrey, Duke Math. J. vol. 6 (1940) pp. 170–215) *with $d\alpha = \delta\rho = 0$; if $\omega \in \mathfrak{p}_2$, then $\delta\alpha$ and $d\beta$ are also in* $\mathfrak{p}_2$. The methods and results are extended to orientable manifolds of class C'-Lip (g_{ij} Lipschitzian). Harmonic forms are found to be Hölder-continuous throughout; further differentiability properties of H, α, and β are proved. (Received January 13, 1955.)

417. L. A. Rubel: *An extension of Carlson's theorem.*

For $0 < \alpha \leq \pi$ the Carlson class $C(\alpha)$ is defined as the class of all entire functions $f(z)$ satisfying, for some $a < \infty$, $c < \alpha$, and all $\epsilon > 0$, the growth restriction: $f(z) = O(1) \exp(a|x| + c|y| + \epsilon|z|)$. A set A of positive integers is said to be total for $C(\alpha)$ if the only function $f \in C(\alpha)$ such that $f(n) = 0$ for all $n \in A$ is the null function $f(z) \equiv 0$. The celebrated theorem of Carlson states that the set I of all positive integers is total for $C(\pi)$. The principal result of the present paper is that a set A is total for $C(\pi)$ if and only if $\overline{D}(A) = 1$ where $\overline{D}(A)$ is the upper asymptotic density of A defined by: $\overline{D}(A) = \lim \sup A(t)/t$ as $t \to \infty$. Here $A(t)$ denotes the number of integers $n \in A$ such that $n \leq t$. For any $\alpha < \pi$ the condition $\overline{D}(A) \geq \alpha/\pi$ is known to be necessary but not sufficient that A be total for $C(\alpha)$. (Received January 10, 1955.)

418t. J. B. Serrin: *A characterization of regular boundary points for second order linear elliptic differential equations.*

Consider the elliptic equation (*), $L(u) = \sum a_{ik} u_{ik} + \sum b_i u_i + cu = f$, where the coefficients are Hölder-continuous functions of $x = (x_1 \cdots x_n)$ in E_n, $c \leq 0$, and $u_{ik} = \partial^2 u/\partial x_i \partial x_k$, $u_i = \partial u/\partial x_i$. Let T be a bounded domain in E_n with boundary t, and let continuous values of u be assigned on t. Then exactly as for the Laplace equation one can associate to each such set of assigned values a generalized solution of the corresponding Dirichlet problem for T (see, for example, Schauder, Math. Zeit. vol. 38 (1934)). Let the assigned values be denoted by ϕ and the associated solution by $u_\phi(x)$, $x \in T$; if a point p of t has the property that, for all ϕ, $\lim_{x \to p} u_\phi(x) = \phi(p)$, then p is said to be regular (for T) with respect to (*). We show that *a boundary point of T is regular with respect to (*) if and only if it is regular with respect to the Laplace equation.* It follows that regular points for the Dirichlet problem for (*) are characterized by Wiener's series criterion (special cases of this result, namely when $L(u)$ has an adjoint and when $a_{ik} = \delta_{ik}$, have already been obtained by Püschel, ibid. vol. 34, and Tautz, ibid. vol. 39). The proof given here, which is somewhat simpler than these earlier ones, makes essential use of the concept of a "barrier." (Received January 12, 1955.)

419t. V. L. Shapiro: *On the integral representation of continuous periodic functions.*

Let $F(x)$, where $x = (x_1, \cdots, x_n)$, be a continuous periodic function of period 2π in each variable in Euclidean n-space, $n \geq 2$, and designate the integral mean of F on the surface of the n-dimensional sphere with radius t and center x by $L(F; x; t)$. Also let T_n designate the torus $\{x = (x_1, \cdots, x_n); -\pi < x_i \leq \pi, i = 1, \cdots, n\}$ and let Z be a closed set of capacity zero contained in T_n. Suppose that $\lim \sup_{t \to 0} |L(F; x; t)|/t^2 < \infty$ in $T_n - Z$ and that $\lim_{t \to 0} 2n[L(F; x; t) - F(x)]/t^2$ is equal to $f(x)$ almost every-

where in T_n where f is an integrable on T_n. Then with the help of the Green's function introduced by Bochner for T_n, namely $G(x) \sim - \sum_{m \neq 0} e^{i(m,x)}/(m, m)$ where m is a lattice point and (m, x) is the scalar product, an integral representation for $F(x)$ in terms of $f(x)$ is obtained. In particular it is shown that for almost all x in T_n, $F(x) = (2\pi)^{-n} \int_{T_n} G(x-y) f(y) dy + (2\pi)^{-n} \int_{T_n} F(x) dx$. The proof of this result makes use of the integral representation obtained previously by Rudin for continuous functions which are not necessarily periodic in terms of their generalized Laplacians. This new integral representation of continuous periodic functions obtained here enables one to prove the following theorem in the uniqueness of multiple trigonometric series: Let the multiple trigonometric series $S = \sum_m a_m e^{i(m,x)}$ be $(C, 1)$ spherically summable to $f(x)$ almost everywhere in T_n where $f(x)$ is integrable on T_n and suppose that $\sum_{m \neq 0} a_m e^{i(m,x)}/(m, m)$ is the Fourier series of a continuous periodic function. Also suppose that $\lim \sup_{R \to \infty} |\sigma_R(x)| < \infty$ in $T_n - Z$ where $\sigma_R(x)$ is the $(C, 1)$ spherical mean of rank R of S. Then S is the Fourier series of $f(x)$. (Received January 6, 1955.)

420. C. J. Standish: *On the representation of a function as a Poisson transform*. I.

Necessary and sufficient conditions are obtained for a function to be represented in the form $f(x) = (1/\pi) \int_{-\infty}^{+} d\beta(t)/1 + (x-t)^2$ where $\beta(t)$ is (a) of bounded variation and (b) monotone and bounded, and in the form $f(x) = (1/\pi) \int_{-\infty}^{+\infty} g(t) dt/1 + (x-t)^2$ where $g(t)$ is Lebesgue integrable on $(-\infty, +\infty)$. The conditions are phrased in terms of a linear differential operator of infinite order employed by H. Pollard (Trans. Amer. Math. Soc. to appear) in constructing a real inversion formula for the Poisson transform. (Received January 6, 1955.)

421. Kurt Strebel: *On the maximal dilation of quasiconformal mappings*.

Let $w(z)$ be a quasiconformal mapping of G_z onto G_w, i.e. topological with finite maximal dilation $K = \sup_Q m'/m$, where Q is any quadrilateral in G_z, m its module, and m' the module of its image in G_w. It is shown that $w(z)$ is absolutely continuous on almost every line $y = ax + b$ for arbitrarily fixed a $(z = x + iy)$. If E is any closed set in G_z and K_0 denotes the maximal dilation of $w(z)$ outside the set E, then $K_0 = K$ if E is of two-dimensional measure zero. If $w(z)$ is only known to be topological in G_z with finite maximal dilation K_0 outside E, then the maximal dilation of $w(z)$ in G_z is equal to K_0 if E is a closed set of Σ-finite linear measure or the union of such a set with a function theoretic nullset 0_{AD}. (Received January 12, 1955.)

422t. John Wermer: *Polynomial approximation on an arc in the space of three complex variables*.

We construct an arc Γ in the space of 3 complex variables z_1, z_2, z_3 with the following property: Not every continuous function on Γ is uniformly approximable on Γ by polynomials in z_1, z_2, z_3. In fact any approximable function maps Γ onto a Peano curve. It follows from this that the Banach algebra of all complex-valued continuous functions on the unit interval has a proper closed subalgebra which separates points and contains the constant 1. (Received January 13, 1955.)

APPLIED MATHEMATICS

423t. G. A. Baker, Jr.: *Equivalence theorem for quantum mechanics*.

Integral formulas are derived for the Hermitian operators of the Schrödinger

formulation of quantum mechanics, thereby giving the operators precise meaning. It is then shown that the Schrödinger formulation is completely equivalent to a formulation based on a quasi-probability distribution of the canonical coordinates and momentum describing the corresponding classical system. The distribution is shown to satisfy an integro-differential equation which corresponds to Schrödinger's wave equation. The quasi-probability distribution is not a probability distribution in the usual sense of the word as it is not in general everywhere non-negative. The transformation theory of this representation of quantum mechanics is investigated. In this paper there has been no need for the customary recourse to the solution of eigenvalue equations in complete orthonormal sets or infinite matrices; however, extensive use has been made of Fourier transform methods. (Received January 6, 1955.)

424t. H. W. Becker: *Diophantine parallel circuits and unbalanced Wheatstone bridges.*

An efficient algorithm for producing all Diophantine circuits, s.p. or bridge, is $t=(wA+B)/(wC-D)=\{A+(BC+AD)/(wC-D)\}/C=(A+F^2/K)/C$. Where $R=t\|w$, it yields the classic results, Dickson's *History* II, pp. 688–691. All solutions are tabulated through $R=155$. Where $R=t\|r\|w$ and $C=r-R$, $t=(Rr+R^2r^2/K)/C$ and $w=(Rr+K)/C$. The procedure is to run through all $1\leqq C\leqq 2R$. There are always Diophantine solutions for $C=1, 2, 3, 5$, but none for $C=4$, $R\equiv 1$ mod 12. All sols. are tabulated through $R=12$. The Wheatstone bridge $R={}'_t r''_u=D/N$, $D=r(t+u)(v+w)$ $+tu(v+w)+vw(t+u)$, $N=r(t+u+v+w)+(t+v)(u+w)$, Amer. Math. Monthly vol. 54 (1947) p. 599. Then $t=\{(R-u)(r+v)+(Rr+uv)^2/K\}/(S-R)$, $w=\{(R-v)$ $\cdot(r+u)+K\}/(S-R)$, $S=u+r+v+w$. The procedure is to run through all compositions of $S\geqq 3$ with $2\leqq R\leqq S-1$. The condition for solutions symmetrically disposed about the median is $A\equiv D$ mod C. The only $DUWB$ whose 5 branches are consecutive integers is ${}^2_b 3^6_4=4$. The $DUWB$ of smallest branches and R all different is apparently ${}^{12}_7 2^{13}_3=4$. R. M. Foster (communication of March 16, 1948) proposed the problem of $DUWB$ whose 6 values are different primes, and supplied the example ${}^{29}_7 71^{19}_{11}=13$. In an iterative Foster bridge, $R=$a branch value. The smallest IFB (calling unity a prime) is ${}^{13}_b 3^7_2=3$. The smallest FB is apparently ${}^3_{13} 2^{19}_5=7$. There's nothing corresponding to FB in DPC. Let $E_n(R)$ or $e_n(R)$, $E_W(R)$ or $e_W(R)$ enumerate the n-branch DPC and $DUWB$ of joint or branch resistance R. Then $E_n(R)>e_n(R)$; but $e_W(R)=\infty$ $\neq E_W(R)$, e.g. $E_W(2)=9$. $E_W(R, S)$ is tabulated through $S=12$: 2152 $DUWB$. (Received December 3, 1954.)

425t. Jim Douglas, Jr.: *The solution of the diffusion equation by a high order correct difference equation.*

The numerical solution of $u_{xx}=u_t$, $0<x<1$, $t>0$, subject to $u(x, 0)=f(x)$, $u(0, t)$ $=g(x)$, $u(1, t)=h(t)$ has been treated by means of several difference equations. Among the implicit methods used have been the backwards difference equation and the Crank-Nicolson equation. It is well known that the error in either case is $O(\Delta t)$, if $\lambda=4\Delta t/(\Delta x)^2$ is held constant as $\Delta t\to 0$. The modification of the Crank-Nicolson equation is proposed: $(1-2/3\lambda)\Delta_x^2 w_{i,n+1}+(1+2/3\lambda)\Delta_x^2 w_{in}=2(w_{i,n+1}-w_{in})/\Delta t$, where $w_{in}=w(i\Delta x, n\Delta t)$ and $\Delta_x^2 w_{in}=(w_{i+1,n}-2w_{in}+w_{i-1,n})/(\Delta x)^2$. The above difference equation is high order correct not only in the t-direction as is the Crank-Nicolson but also in the x-direction. It is shown that, for any $\lambda>0$, the error is $O((\Delta t)^2)$ as $\Delta t\to 0$. The method is quite practical, as it involves many fewer steps to reach a given time with a specified maximum allowable error and only twice as much work per time step as the forward difference equation. (Received January 10, 1955.)

426. Arthur Erdélyi (p), Maurice Kennedy, J. L. McGregor, and C. A. Swanson: *Asymptotic forms of Coulomb wave functions.* Preliminary report.

Asymptotic representations are obtained for the regular and irregular Coulomb wave functions, $F_L(\rho, \eta)$ and $G_L(\rho, \eta)$, and for their derivatives. It is assumed that L is fixed, $\eta \rightarrow +\infty$, and ρ/η is positive and unrestricted. With one exception (to be noted below) the asymptotic forms are expressed in terms of Airy integrals and their derivatives. The asymptotic forms are obtained by a comparison of differential equations, and the constants are determined by means of the leading terms in the known expansions of F_L and G_L for small and large ρ: the integral representations of Coulomb wave functions are not used. Asymptotic representations holding uniformly in the respectively ranges of ρ/η are obtained as follows. For G_L, there is a single asymptotic form for $\rho/\eta \geq \epsilon > 0$, and this form remains valid if η is fixed and $\rho \rightarrow \infty$. For F_L, there is an asymptotic form for $\rho/\eta \geq \epsilon > 0$; there is a second asymptotic form for $\rho/\eta \geq 2$, and this second form remains valid if η is fixed and $\rho \rightarrow \infty$; and there is a third asymptotic form, in terms of modified Bessel functions of order $2L+1$, for $0 \leq \rho/\eta \leq 2 - \epsilon < 2$, and this form remains valid if η is fixed and $\rho \rightarrow 0$. (Received January 13, 1955.)

427t. Isadore Heller: *Geometric characterization of cyclic permutations.*

P is the set of all n by n permutation matrices p, q, $\cdots$ which are interpreted as points in n^2-space. The convex hull C of P is a polyhedron of dimension $(n-1)^2$ and has P as the set of vertices. The note proves: 1. Two permutations p and q are neighbors of order k—that is, they are on the same k-dimensional face of C and have no faces of lower dimension in common—if and only if $p^{-1}q$ is a product of k disjoint cycles. 2. (Corollary) The range of k consists of all integers in the closed interval $[0, n/2]$. 3. (Corollary) The cyclic permutations are the order-one-neighbors of the identity permutation. Cycles of length s (i.e. leaving $n-s$ elements fixed) have distance $(2s)^{1/2}$ from the identity. Geometrically the set of order-one-neighbors of any permutation is equivalent to the set of cyclic permutations. For related results see T. S. Motzkin, *The assignment problem*, Proc. 6th Appl. Math. Symp., McGraw-Hill, 1955. (Received January 10, 1955.)

428t. Jacob Korevaar: *Distributions and their Laplace transforms defined from the point of view of applied mathematics.*

Define $f^{(-1)}(t) = \int_0^t f(u)du$, $f^{(-2)}(t) = \int_0^t f^{(-1)}(u)du$, etc. A sequence of integrable functions $\{f_n(t)\}$, $n = 1, 2, \cdots$, is called fundamental if for some positive integer p the sequence $\{f_n^{(-p)}(t)\}$ is uniformly convergent on every finite interval $(0, A)$. A fundamental sequence defines a distribution ϕ on the half-line $t \geq 0$. An integrable $f(t)$ is identified with the distribution $[f]$ defined by $f_n(t) \equiv f(t)$, $n = 1, 2, \cdots$. For $\tau \geq 0$ the sequence $f_n(t) = 0$ ($0 < t < \tau$, $t > \tau + 1/n$), $f_n(t) = n(\tau < t < \tau + 1/n)$ defines the distribution δ_τ (δ-"function" or unit impulse "function"). Equality of distributions is defined, and it is shown that every distribution ϕ may be redefined by a fundamental sequence of continuous functions $h_n(t)$ which have a piecewise continuous derivative for $t \geq 0$ while $h_n(0) = 0$. The derivative ϕ' of ϕ is then defined by the fundamental sequence $\{h_n'(t)\}$. Examples. $\delta_\tau = U_\tau'$, where U_τ is the unit step function. $[f]' = [f'] + f(0)\delta_0$ when f is continuous, f' piecewise continuous. A distribution ϕ is said to be of exponential type k if it may be given by a fundamental sequence $\{f_n(t)\}$ such that $f_n(t) = 0$ for $t > c_n$ while for some M, k and $q \geq 0$, $|f_n^{(-q)}(t)| \leq Me^{kt}$ ($n = 1, 2, \cdots$; $0 \leq t < \infty$). The

Laplace transform $L(\phi)$ of a distribution $\phi\colon \{f_n(t)\}$ of exponential type k is defined as $L(\phi) = \lim_{n\to\infty} L(f_n)$, $L(f_n) = \int_0^\infty f_n(t)e^{-st}dt$ (Re $s > \max(k, 0)$). Thus $L(\delta_r) = se^{-rs}$. One has $L(\phi') = sL(\phi)$. A uniqueness theorem is proved, and applications are made to differential equations. (Received January 10, 1955.)

GEOMETRY

429. C. C. Hsiung: *A theorem on surfaces with a closed boundary.*

Let S (S^*) be an orientable surface of class C''' in a three-dimensional Euclidean space with positive Gaussian curvature and a closed boundary C (C^*). Suppose that there is a one-to-one correspondence between the points of S and S^* such that the third fundamental forms and the sums of the principal radii of curvature of S and S^* at corresponding points are respectively equal. It is proved that if the two boundaries C and C^* are congruent, then S and S^* are congruent or symmetric. For closed S and S^*, this is a classical theorem of E. B. Christoffel. (Received February 21, 1955.)

430t. Shôshichi Kobayashi: *Holonomy groups of hypersurfaces.*

The author obtained the following theorem: the homogeneous holonomy group of an n-dimensional compact orientable Riemannian manifold M, imbedded isometrically in an $n+1$ dimensional Euclidean space E, is the entire proper orthogonal group $SO(n)$. The proof is based on the following facts: (1) the Gauss spherical mapping $\tilde{h}$ of M into an n-dimensional unit sphere S induces a bundle map h of the bundle of orthogonal frames on M into the bundle of orthogonal frames on S and the Riemannian connection on M can be induced by $\tilde{h}$ from the Riemannian connection on S; (2) if M is compact, there is at least one point x_0 of M such that h is a homeomorphism of a neighborhood U of x_0 onto $h(U)$; (3) the local homogeneous holonomy group of M at x_0 is equal to the local holonomy group of S at $h(x_0)$; (4) the local homogeneous holonomy group of S is $SO(n)$ at every point of S. The theorem is true, even if M is non-orientable: the restricted homogeneous holonomy group of M is $SO(n)$. As an immediate consequence, it can be proved that the restricted nonhomogeneous holonomy group of M is the group of all proper motions in an n-dimensional Euclidean space. Another consequence is that it is impossible to imbed isometrically an n-dimensional compact pseudo-Kaehlerian manifold into an $n+1$-dimensional Euclidean space. (Received December 13, 1954.)

431t. J. F. Nash: *Extension of the solution of the imbedding problem to open Riemannian manifolds.*

By means of a simple device one can show that an open Riemannian manifold can be imbedded isometrically in Euclidean space if this can be done for spheres. If an n-dimensional sphere generally requires $N(n)$ dimensions, then any n-manifold can be imbedded in $E^{(n+1)N(n)}$, and with the same differentiability as for the spheres, up to C^∞. An open n-disk can be mapped onto the complement of a single point on an n-sphere so that the mapping is C^∞ and has all derivatives zero at the periphery. Then if the sphere has a nonsingular positive metric this corresponds to a metric on the disk that goes to zero in a C^∞ manner at the periphery. One can cover a C^∞ manifold (and any differentiable manifold can be regarded as a C^∞ manifold) by a collection of C^∞ images of the open n-disk in such a way that there are $n+1$ classes of images and no two of the same class meet. Then with an appropriate procedure it is easy to assign a metric to each disk image that corresponds to a nonsingular positive metric

on the sphere which, with a point deleted, corresponds to the disk. And this can be done so that the metric tensors on the disk images (and the spheres) have the same differentiability as the metric tensor on the manifold, with it being their sum. Now one imbeds each n-sphere isometrically in $E^{N(n)}$ and arranges for the point that is deleted to be the origin. Then these imbedding functions correspond to good functions on the manifold that are zero outside of the corresponding disk image. If one uses the same imbedding functions within each of the $n+1$ classes of disk images one obtains an isometric imbedding of the manifold in $E_1^{N(n)} \times E_2^{N(n)} \times \cdots \times E_{n+1}^{N(n)} = E^{(n+1)N(n)}$. For 2-manifolds it is possible one might arrange that the induced metric on the spheres have positive curvature everywhere and then use the work that has been done on the problem of Weyl to get an imbedding, perhaps in 9-space. (Received January 17, 1955.)

432. Valdemars Punga: *The concept "composite space" and its differential geometry.*

The author defines the composite space as a set of objects called elements, where each object consists of an n-dimensional point $(x^1, x^2, \cdots, x^n)$ associated with r tensors ${}^i p_{\beta_1 \cdots \beta_p}^{\alpha_1 \cdots \alpha_m}$, $i = 1, 2, \cdots, r$. The most investigated example of a composite space is the so-called Cartan space of line elements (x^α, p^α), where $(x^\alpha, cp^\alpha) = (x^\alpha, p^\alpha)$, $c > 0$, and p^α is a contravariant vector. The Kawaguchi space is also an example of a composite space whose elements are $(x^\alpha, {}^1 p^\alpha, {}^2 p^\alpha, \cdots, {}^r p^\alpha)$, where ${}^i p^\alpha$ are r pseudo-vectors not completely arbitrary, but satisfying some differential equations. The König space is an example of a composite space, where the point and the associated tensors are of different dimensions. Also we can associate with an n-dimensional point any geometric object (not necessarily a tensor) and consider this combination as an element of a composite space. The author discusses the differential geometry of such spaces in general and in particular the composite space $(x^\alpha, p^\alpha, w_\alpha)$, where p^α and w_α are contra- and covariant vectors respectively. The concept of a composite space is in particular very convenient for the development of Finsler geometry and its generalizations. (Received January 12, 1955.)

STATISTICS AND PROBABILITY

433. Jerome Blackman: *On the approximation of a distribution function by an empiric distribution.*

Let $x_1, x_2, \cdots, x_n$ be independent random variables with the common distribution function $F(x)$ and the empiric distribution function $F_n(x)$. Let a_n be the value which minimizes $\int_{-\infty}^{\infty} |F(x-a) - F_n(x)|^2 dF^{(x-a)}$. The asymptotic distribution of $n^{1/2} a_n$ is obtained under the hypothesis that $F(x)$ has three continuous and bounded derivatives. (Received December 20, 1954.)

434t. K. L. Chung and C. Derman: *Non-recurrent random walks.*

A necessary and sufficient condition is proved for every infinite set of states in a (denumerable) Markov chain (with stationary transition probabilities) to be visited infinitely often with probability one. In the case of sums of independent, identically distributed, integer-valued random variables with span 1 every infinite set of positive (negative) integers is visited infinitely often with probability one if the mean is finite and positive (negative). A similar result is given when the common distribution is absolutely continuous. (Received January 17, 1955.)

435. J. M. Shapiro: *On sums of truncated random variables.*

Let (x_{nk}) $(k=1, 2, \cdots, k_n; n=1, 2, \cdots)$ be a double sequence of infinitesimal random variables such that for each n, $x_{n1}, \cdots, x_{nk_n}$ are independent. Let $S_n = x_{n1} + \cdots + x_{nk_n}$ and let $F_n(x)$ be the distribution function of S_n. Let $F(x)$ be any infinitely divisible distribution and let $G(x)$ be the bounded nondecreasing function determined by the Levy-Khintchine formula for the representation of the characteristic function of $F(x)$. For any $a > 0$ ($\pm a$ a continuity point of $G(x)$) let x_{nk}^a be defined by $x_{nk}^a = x_{nk}$ if $|x_{nk}| \leqq a$, $x_{nk}^a = 0$ otherwise, and let $F_n^a(x)$ be the distribution function of $S_n^a = x_{n1}^a + \cdots + x_{nk_n}^a$. It is shown that if $F_n^a(x) \to F(x)$ it is necessary that (1): $G(x) = G(+\infty)$ for $x \geqq a$ and (2): $G(x) = G(-\infty)$ for $x \leqq -a$ (whether $F_n(x) \to F(x)$ or not); and if $F_n(x) \to F(x)$ then (1) and (2) are also sufficient for $F_n^a(x) \to F(x)$. In particular if $F(x)$ is the Poisson distribution and if $F_n(x) \to F(x)$, then for any $a > 1$, $F_n^a(x) \to F(x)$. In this case it is also shown that the variances of $F_n^a(x)$ approach the variance of $F(x)$. (Received January 12, 1955.)

TOPOLOGY

436t. R. L. Blair: *A note on lattices of continuous functions.* Preliminary report.

If X is a topological space and K is a chain endowed with its order topology, then $C(X, K)$ denotes the set of all continous functions from X to K. Kaplansky (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 617–623) has shown that if X is compact Hausdorff then, as a lattice, $C(X, R)$ characterizes X (here R denotes the real numbers). On the other hand (cf. Gelfand and Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 11–15) if X is completely regular then, as a ring, $C(X, R)$ characterizes the Stone-Čech compactification $\beta(X)$ of X. These results are here subsumed in the following theorem. If X is completely regular then, as a lattice alone, $C(X, R)$ characterizes $\beta(X)$. A key step in the proof is a sharpened version of Kaplansky's result (loc. cit., Theorem 1) which may be formulated as follows: If α, $\beta \in K$, write $\alpha \ll \beta$ in case $\alpha \leqq \beta$ and in fact $\alpha < \beta$ unless α is maximal or β is minimal in K. A sublattice B of $C(X, K)$ *separates* X in case for any f, $g \in B$ and any disjoint closed sets F, G of X there is an $h \in B$ with $h(x) \ll f(x)$, $g(y) \ll h(y)$ for all $x \in F$, $y \in G$. Then if X_i is compact Hausdorff and if B_i is a sublattice of $C(X_i, K)$ which separates X_i ($i = 1, 2$), any lattice isomorphism between B_1 and B_2 induces a homeomorphism between X_1 and X_2. (Received January 14, 1955.)

437t. L. M. Blumenthal and V. L. Klee: *On metric independence and linear independence.*

Suppose M is a metric space with distance function ρ, and for each $x \in M$ let f_x be the function $\rho(x, y) \mid y \in M$. A subset A of M is said to be "metrically dependent in M" provided the set of functions $\{f_a : a \in A\}$ is linearly dependent over M; otherwise, A is "metrically independent in M." Results of I. J. Schoenberg, A. H. Stone, and C. H. Dowker are used to show that every metric space has a bounded homeomorph in which every subset A is metrically independent in A. From this follows: (I) If M is a metric space, then M is homeomorphic with a linearly independent subset of CM, the space of all bounded real continuous functions on M. (II) Every separable metric space is homeomorphic with a linearly independent subset of $C[0, 1]$. (With the words "linearly independent" omitted, (I) and (II) are well known.) It is proved also that a metric quadruple Q is metrically dependent in Q if and only if Q is pseudo-linear. (Received December 13, 1954.)

438. J. F. Nash: *A path space and the Stiefel-Whitney classes.*

A very simple proof of the topological invariance of the Stiefel-Whitney classes of a differentiable manifold may be obtained as a direct corollary of a general theorem on a bundle of paths. Consider the class of all parametrized paths in a manifold which have the property of never returning to their initial points. Using the initial point to define the projection mapping to the manifold, this becomes a fibre space of the manifold. By a direct, and quite elementary, construction of the two appropriate mappings this can be rather easily seen to be homotopically equivalent as a fibre space to the fibre space of tangents. Since the Stiefel-Whitney classes depend only on the fibre homotopy type of the tangent bundle, and since the path fibre space is obviously topologically invariant, their topological invariance follows. (Received January 13, 1955.)

439t. L. E. Ward, Jr.: *Fixed point theorems for special types of continua.*

In a previous paper (Bull. Amer. Math. Soc. Abstract 60-5-632) the author has defined a *tree* to be a continuum (=compact connected Hausdorff space) in which any two points are separated by some third point. R. H. Bing has defined a *tree-like continuum* (Duke Math. J. vol. 18, p. 653). The tree-like continua include the dendrites (metric trees) and certain indecomposable continua. A space has f.p.p. if it has the fixed point property for continuous functions. The following theorems are proved. I. A tree has f.p.p. II. A tree-like continuum has f.p.p. III. If X is a continuum which admits a continuous linear order, if I denotes the closed unit interval, and if T is a separable space such that $I \times T$ has f.p.p., then $X \times T$ has f.p.p. II settles a question raised by Bing. I and II are separate generalizations of the fixed point theorem for dendrites and are proved by order-theoretic methods. (Received December 1, 1954.)

440t. Chien Wenjen: *Quasi-equicontinuous sets of functions.*

A sequence of functions $\{f_n(x)\}$ from a topological space X to a metric space Y is said to be ϵ-related if for any $\epsilon > 0$ there exists a neighborhood $U(x)$ of x such that for any point x' in $U(x)$ there is a number $N_\epsilon(x, x')$ satisfying: $\rho[f_n(x), f_n(x')] < \epsilon$ if $n > N_\epsilon(x, x')$. A family of continuous functions F from the space X to the space Y is said to be quasi-equicontinuous if in every infinite subset Q of F and at any point $x \in X$ there is a sequence contained in Q which is ϵ-related for any $\epsilon > 0$. Theorem: If X is a locally separable space and Y is metric, a set of functions $F \subset Y^X$, where Y^X denotes the set of all continuous functions from X to Y, is compact under p-topology if and only if (1) F is closed in Y^X, (2) $F(x) = \cup_{f \in F} f(x)$ is compact for any $x \in X$, (3) F is quasi-equicontinuous. An immediate consequence of the theorem is: Let F be a family of continuous functions from a separable space X to a metric space Y. The necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any sequence of functions of F are: (1) $F(x)$ is countably compact for any $x \in X$, (2) F is quasi-equicontinuous. (Received December 30, 1954.)

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