ON THE CHARACTERS OF A SEMISIMPLE LIE GROUP

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Let G be a connected semisimple Lie group and let Z denote its center. If π is a representation [2c] of G on a Hilbert space $\mathfrak{H}$ we consider the space V consisting of all finite linear combinations of elements of the form

$$\int f(x)\pi(x)\psi dx \qquad (f \in C_c^\infty(G),\ \psi \in \mathfrak{H}),$$

where dx is the Haar measure of G and $C_c^\infty(G)$ is the set of all (complex-valued) functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. V is called the Gårding subspace of $\mathfrak{H}$. Let R and C be the fields of real and complex numbers respectively and $\mathfrak{g}_0$ the Lie algebra of G. We complexify $\mathfrak{g}_0$ to $\mathfrak{g}$ and denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}$ [2a]. Then there exists a (uniquely determined) representation π_V of $\mathfrak{B}$ on V such that $\pi_V(X)\psi = \lim_{t\to 0} (1/t)\{\pi(\exp tX)\psi - \psi\}$ $(X \in \mathfrak{g}_0,\ \psi \in V,\ t \in R)$. Let $\mathfrak{Z}$ denote the center of $\mathfrak{B}$. We say that π is quasi-simple if there exist homomorphisms η and χ of Z and $\mathfrak{Z}$ respectively into C such that $\pi(\zeta)\phi = \eta(\zeta)\phi$, $\pi_V(z)\psi = \chi(z)\psi$ for all $\zeta \in Z$, $z \in \mathfrak{Z}$, $\phi \in \mathfrak{H}$ and $\psi \in V$. η is then called the central character and χ the infinitesimal character of π. An irreducible unitary representation is automatically quasi-simple [5].

Let A be a bounded linear operator on $\mathfrak{H}$. We say that A is of the trace class or A has a trace if for every complete orthonormal set $(\psi_j)_{j\in J}$ in $\mathfrak{H}$ the series[1] $\sum_{j\in J} (\psi_j, A\psi_j)$ converges absolutely and its sum is independent of the choice of the complete orthonormal set.[2] We call this sum the trace of A and denote it by Sp A. Now suppose π is quasi-simple and irreducible. Then it can be shown (see [2e]) that for any $f \in C_c^\infty(G)$ the operator $\int f(x)\pi(x)dx$ is of the trace class. If we denote its trace by $T_\pi(f)$ we get a linear function T_π on $C_c^\infty(G)$ which is actually a distribution (see [4; and 2e]). We call this distribution the character of π. Our object is to try to determine T_π.

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[1] As usual (ϕ, ψ) denotes the scalar product of the two elements ϕ and ψ in $\mathfrak{H}$.

[2] Actually it can be shown that this independence of the sum follows automatically from the absolute convergence of the series for every orthonormal base.

Let $x \rightarrow \text{Ad}(x)$ $(x \in G)$ denote the adjoint representation of G. If λ is an indeterminate and I is the identity mapping of $\mathfrak{g}_0$ we consider the characteristic polynomial $P_x(\lambda) = \det(\lambda I - \text{Ad}(x))$ of $\text{Ad}(x)$. Let l be the highest integer such that $(\lambda-1)^l$ divides $P_x(\lambda)$ for every $x \in G$. We expand $P_x(\lambda)$ in powers of $(\lambda-1)$ and denote by $D(x)$ the coefficient of $(\lambda-1)^l$ in this expansion. Then it is clear that D is an analytic function on G which, in view of our definition of l, cannot be identically zero. The integer l is called the rank of G. Let S denote the set of all $x \in G$ for which $D(x)=0$. Then S is a closed, nowhere dense subset of G and its complement G' is open and everywhere dense in G. An element $x \in G$ is called singular or regular according as $x \in S$ or $x \in G'$. Since $\text{Ad}(zx) = \text{Ad}(x)$ $(z \in Z)$ it is obvious that $ZS = S$ and $ZG' = G'$. Also since $P_{yxy^{-1}}(\lambda) = P_x(\lambda)$ $(x, y \in G)$, it follows that $D(yxy^{-1}) = D(x)$ and therefore $ySy^{-1} = S$, $yG'y^{-1} = G'$ $(y \in G)$.

When speaking of a real differentiable (or analytic) manifold M let us agree to include the case when M is not connected but the various connected components of M, which are all manifolds in the usual sense, have the same dimension. Under this definition every open subset U of M is again a manifold. Let $C_c^\infty(U)$ denote the set of all complex-valued functions on M which are indefinitely differentiable and which vanish outside some compact subset of U. In particular suppose U is an open subset of G and F is a (complex-valued) function on U. We say that F is locally summable if it is summable on every compact subset of U with respect to the Haar measure of G. T being a distribution on G we say that $T = F$ on U if F is locally summable and

$$T(f) = \int f(x)F(x)dx$$

for all $f \in C_c^\infty(U)$. Our main result may now be stated as follows.

THEOREM 1. *Let π be an irreducible quasi-simple representation of G on $\mathfrak{H}$ and let T_π denote its character. Then there exists an analytic function F_π on G' such that $T_\pi = F_\pi$ on G'.*

Although in general G' is not connected, there always exist a finite number of connected components $G_1, \cdots, G_r$ of G' such that $ZG_i \cap ZG_j = \emptyset$ if $i \neq j$ and $G' = \bigcup_{i=1}^r ZG_i$. Moreover if η_π is the central character of π it is easy to show that $F_\pi(zx) = \eta_\pi(z)F_\pi(x)$ $(z \in Z, x \in G')$. Hence the knowledge of F_π on $G_1 \cup G_2 \cup \cdots \cup G_r$ is sufficient to determine it completely. On the other hand it is possible to give examples in which F_π vanishes everywhere on one of the components G_i without being zero identically on G'. However in case G is either a compact or a complex group, G' is connected and then F_π is completely determined by its restriction on any nonempty open subset of G'.

For any $x \in G$ and $f \in C_c^\infty(G)$ we define the function f^x by $f^x(y) = f(x^{-1}yx)$ $(y \in G)$. Then $f^x \in C_c^\infty(G)$ and $T_\pi(f^x) = T_\pi(f)$ (see [**2e**]). From this it follows that $F_\pi(xyx^{-1}) = F_\pi(y)$ $(x \in G, y \in G')$. Now let A be a maximal abelian subgroup of G which is not contained in S. (We call such a group a Cartan subgroup of G.) Obviously A is closed in G. Let $A' = A \cap G'$ and $V = \bigcup_{x \in G} xA'x^{-1}$. Then V is an open subset of G' and it is clear from the above remarks that F_π is determined completely on V as soon as we know it on A'. Let $\mathfrak{h}_0$ denote the subalgebra of $\mathfrak{g}_0$ corresponding to A. Then the complexification $\mathfrak{h}$ of $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Let W be the Weyl group (see [**2b**]) of $\mathfrak{g}$ with respect to $\mathfrak{h}$ so that W is a finite group of nonsingular linear transformations of $\mathfrak{h}$. If Λ is a linear function on $\mathfrak{h}$ and $s \in W$ we define the linear function $s\Lambda$ by the rule $s\Lambda(H) = \Lambda(s^{-1}H)$ $(H \in \mathfrak{h})$. Let $(H_1, \cdots, H_l)$ be a base for $\mathfrak{h}$ over C. A function P on $\mathfrak{h}$ is called a polynomial function if there exists a polynomial $p(x_1, \cdots, x_l)$ in l variables $(x_1, \cdots, x_l)$ with coefficients in C such that $P(H) = p(a_1, \cdots, a_l)$ if $H = a_1H_1 + \cdots + a_lH_l (a_i \in C)$. The degree of p is also called the degree of P. Clearly these definitions are independent of the choice of the base $(H_1, \cdots, H_l)$.

THEOREM 2. *Let π and F_π be as in Theorem* 1. *Then there exists a linear function Λ on $\mathfrak{h}$ with the following property. For any $a \in A'$ we can choose polynomial functions p_s $(s \in W)$ on $\mathfrak{h}$ such that*

$$F_\pi(a \exp H) = \left| D(a \exp H) \right|^{-1/2} \sum_{s \in W} p_s(H) e^{s\Lambda(H)}$$

for all H lying sufficiently near zero in $\mathfrak{h}_0$. Λ is unique up to an operation of W and if N is the number of elements σ in W such that $\Lambda = \sigma\Lambda$, the degree of every p_s is necessarily smaller than N.

In particular if $s\Lambda \neq \Lambda$ except when s is the unit element of W, p_s must all be constants. (It is hardly necessary to point out the resemblance of the above formula to the one given by Weyl [**6**] for the irreducible characters of a compact semisimple Lie group. It should also be compared with the results of Gelfand and Naimark [**7**] on the unitary characters of the complex classical groups (see also [**2f**, p. 511])). Although A' is not necessarily connected, it is possible to select a finite set $B_1, \cdots, B_r$ of its connected components such that $A' = \bigcup_{i=1}^r ZB_i$. Therefore in order to determine F_π on A', it is sufficient to know η_π and the restrictions of F_π on some nonempty open subsets of $B_1, \cdots, B_r$. Hence if Λ is known, Theorem 2 gives us a formula for F_π on A' in terms of a finite number of undetermined constants. On the other hand we shall see presently that Λ is completely determined (up to an operation of W) by the infinitesimal character χ_π of π.

Two Cartan subgroups A_1 and A_2 are said to be conjugate if $A_2 = xA_1x^{-1}$ for some $x \in G$. It is always possible to choose a finite number of distinct Cartan subgroups $A_1, \cdots, A_k$ such that every Cartan subgroup is conjugate to exactly one of these. Then if $A_i' = A_i \cap G'$ and $V_i = \bigcup_{x \in G} xA_i'x^{-1}$, G' is the disjoint union of $V_1, \cdots, V_k$. This shows that if η_π and χ_π are known, F_π is completely determined in terms of a finite number of constants.[3]

Now we come to a brief outline of the proof. Let M be a differentiable manifold, Q a linear mapping of $C_c^\infty(M)$ into itself and x_0 a point in M. We say that Q is a differential operator at x_0 if there exists a coordinate system $(t_1, \cdots, t_m)$ valid on an open neighborhood U of x_0 and indefinitely differentiable functions $g_{i_1 i_2 \cdots i_p}$ on U $(1 \leqq i_1, \cdots, i_p \leqq m,\ 0 \leqq p \leqq q)$ such that if $f \in C_c^\infty(U)$, Qf is zero outside U and

$$Qf = \sum_{0 \leqq p \leqq q} \sum_{1 \leqq i_1, \cdots, i_p \leqq m} g_{i_1 i_2 \cdots i_p} \frac{\partial^p}{\partial t_{i_1} \cdots \partial t_{i_p}} f$$

on U. If Q is a differential operator at every point in M we say simply that it is a differential operator (on M). T being a distribution on M and Q a differential operator we define a distribution $Q'T$ as follows:

$$(Q'T)(f) = T(Qf) \qquad (f \in C_c^\infty(M)).$$

In particular if g is an indefinitely differentiable function on M it defines a differential operator $Q: f \rightarrow gf$ $(f \in C_c^\infty(M))$. In this case we write gT to denote $Q'T$ so that $(gT)(f) = T(gf)$. It is clear that the product of two differential operators is again a differential operator and therefore the differential operators form an algebra.

Coming back to G, we note that every $X \in \mathfrak{g}_0$ may be regarded as a differential operator on G as follows:

$$(Xf)(x) = \left\{ \frac{d}{dt} f(x \exp tX) \right\}_{t=0} \qquad (f \in C_c^\infty(G),\ x \in G,\ t \in R).$$

Thus it is easy to see that $\mathfrak{B}$ may be identified in a natural way with a subalgebra of the algebra of differential operators on G. Then for any $b \in \mathfrak{B}$ we have a linear transformation b' of the space of distribu-

[3] Actually it is possible to improve Theorems 1 and 2 and show that T_π coincides with an analytic function on an open subset of G which, in general, is larger than G' and therefore has fewer connected components.

tions on G. Since $(b_1b_2)'=b_2b_1$ $(b_1, b_2\in\mathfrak{B})$ the mapping $b\rightarrow b'$ is an anti-representation of $\mathfrak{B}$. Let ϕ denote the anti-automorphism of $\mathfrak{B}$ over C which is uniquely determined by the condition that $\phi(X)=-X$ $(X\in\mathfrak{g})$. Then $b\rightarrow(\phi(b))'$ $(b\in\mathfrak{B})$ is a representation of $\mathfrak{B}$. T being any distribution on G we now define $bT=(\phi(b))'T$ $(b\in\mathfrak{B})$. Then $(bT)(f)=T(\phi(b)f)$ $(f\in C_c^\infty(G))$. If χ_π is the infinitesimal character of π and $z\in\mathfrak{Z}$, $f\in C_c^\infty(G)$, it is easy to see that

$$\int(\phi(z)f)(x)\pi(x)\psi dx = \pi_0(z)\left(\int f(x)\pi(x)\psi dx\right)$$
$$= \chi_\pi(z)\int f(x)\pi(x)\psi dx$$

for all $\psi\in\mathfrak{H}$. (Here π_0 is the representation of $\mathfrak{B}$ on the Gårding subspace of $\mathfrak{H}$.) From this it follows that $zT_\pi=\chi_\pi(z)T_\pi$ for all $z\in\mathfrak{Z}$. Hence T_π is an eigen-distribution for each differential operator in $\mathfrak{Z}$.

On the other hand let A be a Cartan subgroup of G. Put $A'=A\cap G'$ and $V=\bigcup_{x\in G} xA'x^{-1}$ as before. We regard A' and V as open submanifolds of A and G respectively. Let G^* be the factor space G/A consisting of cosets of the form xA $(x\in G)$. If $h\in A$ and $x^*\in G^*$ we define $h^{x^*}=xhx^{-1}$ where x is any element in the coset x^*. Let dh and dx^* respectively denote the Haar measure on A and the invariant measure on G^*. Then we have the following lemma.

LEMMA 1. *There exists a distribution τ_π on A' with the following property. If $f\in C_c^\infty(V)$, $T_\pi(f)=\tau_\pi(g)$ where g is the function in $C_c^\infty(A')$ given by*

$$g(h) = |D(h)|\int_{G^*} f(h^{x^*})dx^* \qquad (h\in A').$$

Let $\mathfrak{h}_0$ be the Lie algebra of A. Any element $H\in\mathfrak{h}_0$ may be regarded as a differential operator on A so that if $g\in C_c^\infty(A)$,

$$(Hg)(h) = \left\{\frac{d}{dt}g(h\exp tH)\right\}_{t=0} \qquad (h\in A,\ t\in R).$$

Let $\mathfrak{h}$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}_0$ over C and $\mathfrak{U}$ the subalgebra of $\mathfrak{B}$ generated by $(1,\mathfrak{h})$. Then $\mathfrak{U}$ may be identified in a natural way with a subalgebra of the algebra of differential operators on A. For any distribution τ on A' and $u\in\mathfrak{U}$ we define a distribution $u\tau$ on A' as follows:

$$(u\tau)(g) = \tau(\phi(u)g) \qquad (g\in C_c^\infty(A')).$$

Here ϕ is the automorphism of $\mathfrak{U}$ given by $\phi(H)=-H$ $(H\in\mathfrak{h})$.

For every root α of $\mathfrak{g}$ (with respect to $\mathfrak{h}$), choose an element $X_\alpha \neq 0$ in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}$. We introduce some lexicographic order (see [**2b**]) in the set of all roots and denote by P the set of positive roots under this order. Put $\mathfrak{n} = \sum_{\alpha \in P} CX_\alpha$. Then for every $z \in \mathfrak{Z}$ there exists a unique element $\gamma'(z) \in \mathfrak{U}$ such that $z - \gamma'(z) \in \mathfrak{B}\mathfrak{n}$ (see [**2b**, p. 72]). If $2\rho = \sum_{\alpha \in P} \alpha$ there exists a unique automorphism λ of $\mathfrak{U}$ such that $\lambda(1) = 1$ and $\lambda(H) = H - \rho(H)$ $(H \in \mathfrak{h})$. We put $\gamma(z) = \lambda(\gamma'(z))$ $(z \in \mathfrak{Z})$. Let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It is clear that every $s \in W$ defines an automorphism $u \to u^s$ of $\mathfrak{U}$ such that $1^s = 1$ and $H^s = sH$ $(H \in \mathfrak{h})$. An element $u \in \mathfrak{U}$ is called an invariant if $u = u^s$ for all $s \in W$. Let J be the subalgebra of $\mathfrak{U}$ consisting of all invariants. Then (see [**2b**, Lemma 38]) the mapping $z \to \gamma(z)$ $(z \in \mathfrak{Z})$ defines an isomorphism of $\mathfrak{Z}$ onto J. Now let Λ be a linear function on $\mathfrak{h}$. We can extend it uniquely to a homomorphism of $\mathfrak{U}$ into C which takes the value 1 at 1. We agree to denote this extension also by Λ. Then as shown in [**2b**, Theorem 5] we can choose Λ in such a way that $\chi_\pi(z) = \Lambda(\gamma(z))$ for all $z \in \mathfrak{Z}$. Λ is determined up to an operation of W by this condition.

Let $\Delta(h) = |D(h)|^{1/2}$ $(h \in A')$. Then Δ is an analytic function on A' and therefore $\sigma_\pi = \Delta\tau_\pi$ is a well-defined distribution on A'. Now if we transform the differential equation $zT_\pi = \chi_\pi(z)T_\pi$ $(z \in \mathfrak{Z})$ for T_π into a differential equation for σ_π we get the following result which is one of the main steps of our argument.

LEMMA 2. *τ_π being as in Lemma 1 put $\sigma_\pi = \Delta\tau_\pi$. Then*

$$\gamma(z)\sigma_\pi = \chi_\pi(z)\sigma_\pi$$

for every $z \in \mathfrak{Z}$.

Now choose a linear function Λ on $\mathfrak{h}$ such that $\chi_\pi(z) = \Lambda(\gamma(z))$ for $z \in \mathfrak{Z}$. Let ζ be an indeterminate and u an element in $\mathfrak{U}$. Since $\mathfrak{U}$ is abelian we can consider the polynomial

$$\prod_{s \in W} (\zeta - u^s).$$

It is clear that every coefficient of this polynomial lies in J and therefore if w is the order of W, there exist uniquely determined elements $z_1(u), \cdots, z_w(u)$ in $\mathfrak{Z}$ such that

$$\prod\nolimits_{s \in W}(\zeta - u^s) = \zeta^w + \gamma(z_1(u))\zeta^{w-1} + \gamma(z_2(u))\zeta^{w-2} + \cdots + \gamma(z_w(u)).$$

On replacing ζ by u we immediately get the identity

$$u^w + u^{w-1}\gamma(z_1(u)) + u^{w-2}\gamma(z_2(u)) + \cdots + \gamma(z_w(u)) = 0$$

in $\mathfrak{U}$ if we recall that $\mathfrak{U}$ is abelian. Now apply the left side to σ_π and use Lemma 2. Then

$$u^w\sigma_\pi + \chi_\pi(z_1(u))u^{w-1}\sigma_\pi + \cdots + \chi_\pi(z_w(u))\sigma_\pi = 0.$$

But $\chi_\pi(z_j(u)) = \Lambda(\gamma(z_j(u)))$, $1 \leqq j \leqq w$, and since Λ is a homomorphism of $\mathfrak{U}$ into C it is obvious that

$$\prod_{s\in W} (\zeta - \Lambda(u^s)) = \zeta^w + \Lambda(\gamma(z_1(u)))\zeta^{w-1} + \cdots + \Lambda(\gamma(z_w(u))).$$

Therefore the above differential equation for σ_π may be written in the form

$$\prod_{s\in W} (u - \Lambda(u^s))\sigma_\pi = 0.$$

However if $H_1, \cdots, H_l$ is a base for $\mathfrak{h}_0$ over R and $\square = H_1^2 + \cdots + H_l^2$ it is obvious that the differential equation

$$\prod_{s\in W} (\square - \Lambda(\square^s))\sigma_\pi = 0$$

is of the *elliptic* type (see Gårding [**1**]). Hence it follows from the work of Schwartz [**4**, p. 137] and John [**3a**, **3b**] that σ_π must be an analytic function on A'. Now if we take into account Theorem 5 of [**2b**] and the fact that $\prod_{s\in W} (u - \Lambda(u^s))\sigma_\pi = 0$ for every $u \in \mathfrak{U}$, we get Theorem 2 without difficulty.

π being any quasi-simple irreducible representation we denote by T_π, η_π, and χ_π respectively the character, the central character, and the infinitesimal character of π. Also we denote by F_π the analytic function on G' such that $T_\pi = F_\pi$ on G'. Let T be a distribution on G. Since D is an analytic function on G the product D^mT ($m \geqq 0$) is a well-defined distribution. It is then possible to prove the following result.

LEMMA 3. *There exists an integer $m \geqq 0$ with the following property. Suppose $\pi_1, \cdots, \pi_k$ is a finite set of quasi-simple irreducible representations and $c_1F_{\pi_1} + \cdots + c_kF_{\pi_k} = 0$ ($c_i \in C$). Then if $T = c_1T_{\pi_1} + \cdots + c_kT_{\pi_k}$, $D^mT = 0$.*

From this lemma one can deduce the following theorem.

THEOREM 3. *Let π_0 be a quasi-simple irreducible representation of G such that F_{π_0} is not identically zero. Then, apart from infinitesimal equivalence* [**2d**, p. 230], *there exist only a finite number of quasi-simple irreducible representations π such that $F_\pi = F_{\pi_0}$.*

Let $\eta \neq 0$ and $\chi \neq 0$ respectively be given homomorphisms of Z and $\mathfrak{Z}$ into C. Let ω denote the set of all quasi-simple irreducible representations π of G such that $\eta_\pi=\eta$ and $\chi_\pi=\chi$. As we have seen, it follows from Theorems 1 and 2 that the functions F_π $(\pi\in\omega)$ span a finite-dimensional vector space over C. Hence if ω is not empty, we can choose a finite set of elements $\pi_1, \cdots, \pi_k$ in ω such that $F_{\pi_1}, \cdots, F_{\pi_k}$ form a base for this vector space. Then if π is any representation in ω, $F_\pi=c_1F_{\pi_1}+\cdots+c_kF_{\pi_k}$ $(c_j\in C)$. If one could conclude from this equation that $T_\pi=c_1T_{\pi_1}+\cdots+c_kT_{\pi_k}$ it would follow (see [**2e**, Theorem 6]) that π is infinitesimally equivalent to some π_j $(1\leqq j\leqq k)$ and therefore, apart from infinitesimal equivalence, ω has only a finite number of representations. Therefore it is important to consider the following question.

Let $\pi_1, \cdots, \pi_k$ be a finite set of quasi-simple irreducible representations of G such that $c_1F_{\pi_1}+\cdots+c_kF_{\pi_k}=0$ $(c_i\in C)$. Then is it always true that $c_1T_{\pi_1}+\cdots+c_kT_{\pi_k}=0$?

I believe the answer is yes but do not know how to prove it.

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