TOPOLOGICAL ANALYSIS

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1. **Introduction.** Apart from the really spectacular development of the field of topology during the last 50 years, one of the most interesting and satisfying related phenomena has been the widespread interaction between topology and other branches of mathematics. This has occurred in nearly all fields of mathematics. It has been notable in algebra, algebraic geometry, differential geometry and in various types of analysis. However it is the connection with the theory of functions, and particularly functions of a complex variable, which has developed and is being actively developed at present that I should like to discuss in some detail today. As I have used the term, *topological analysis* refers to those results of the analysis type, theorems about functions or mappings from one space onto another or about real or complex valued functions in particular, which are topological or pseudo-topological in character and which are obtainable largely by topological methods. Thus in a word we have analysis theorems and topological proofs. As just indicated however, what I shall say today will be confined largely to results closely related to analytic functions of a complex variable. Since one of the main roots—if indeed not the taproot— of topology rests firmly in the recognition by Riemann and Poincaré of the fundamental and inescapable topological nature of such functions, much of the work to be described represents a return of topology to some of the original situations and problems which motivated its beginnings and to which it owes much for its early development.

Contributions of fundamental concepts and results in this type of work have been made during the past 25 years by a large number of mathematicians. Among these should be mentioned (1) Stoïlow [1], the originator of the interior or open mapping, who early recognized lightness and openness as the two fundamental topological properties of the class of all nonconstant analytic functions; (2) Eilenberg [2] and Kuratowski [3] who introduced and used an exponential representation for a mapping and related it to properties of sets in a plane, (3) Morse and Heins [4], whose studies on invariance of topological indices of a function under admissible deformations of curves in the

complex plane greatly clarified the action of the mapping in the region and near the boundary and opened the way toward admissible simplifying assumptions, (4) the Nevanlinna [5] brothers and L. Ahlfors [6] whose oustanding work on exceptional values of analytic functions lead to conclusions partly topological in character which contain the suggestion of new connections with topology still awaiting development, and (5) Ursell and Eggleston [7] as well as Titus and Young [8] who have contributed elementary proofs for the lightness and openness of analytic mappings using novel methods which have stimulated considerable further effort using these methods in the same area as well as for mappings in a more general topological setting. My own work on this subject began around 1936 as a result of reading some of Stoïlow's early papers, and has been published in an extended sequence of papers spanning the interval of nearly 20 years to the present. On two occasions, however, summaries of some of my results have been given and these are to be found in Memoirs No. 1 of the American Mathematical Society series, entitled *Open mappings on locally compact spaces* and as Lecture No. 1 in the University of Michigan's recently published *Lectures on functions of a complex variable*. In what will be said today an attempt will be made to minimize the overlapping with these earlier lectures—although some will be unavoidable—and concentrate mostly on recent results not previously announced and on some promising areas for new development.

2. **Openness and closedness of mappings.** We begin with a brief discussion of openness and closedness of mappings in a general setting, together with some closely related mapping types. Spaces mentioned will always be assumed to have an open set topology in which the weak separation axiom is satisfied that for any two distinct points x and y there is an open set containing x but not y. Thus we have T_1-spaces. Actually the points to be considered are of primary interest in much more restricted spaces, such as separable metric ones or even Euclidean spaces so that little will be lost in following the discussion if one thinks in terms of these spaces. A mapping $f(X) = Y$, that is, a single valued continuous transformation, of a space X onto a space Y is *open* or *closed* provided the image of every open or closed set in X is open or closed respectively in the space Y.

For example, the mapping of the complex z-plane Z onto the w-plane W generated by the function $w = z^2$ is open since the image of any open set in the z-plane such as a sector is an open set, a larger sector in the w-plane. On the other hand, if the function is modified making it $w = \bar{z}^2$ on the left-hand half-plane and keeping it $w = z^2$ on

the right half-plane, the mapping is no longer open. For the effect now is to fold the left half-plane onto the right half-plane across the y-axis and then stretch the right half-plane so as to cover the whole w-plane by drawing the positive and negative y-axis together along the negative u-axis. Thus, a circular region with center on the positive y-axis would have as its image a semicircular region containing points of the negative u-axis but no points below this axis so that the image is not open.

Similarly, for closedness, any mapping on an ordinary compact metric space is necessarily closed, since all closed sets in such a space are compact and compactness is an invariant property for all mappings. On the other hand the mapping $w = e^z$ of Z into W is not closed because, for example, the set of points on the right arm of the hyperbola $y = 1/x$ is closed in Z whereas its image under the exponential mapping approaches the circle $|w| = 1$ asymptotically but contains no point of this circle. For an even simpler example it is of interest to note that the mapping $x = \cos t$, $y = \sin t$ of the half-open interval $0 \leq t < 2\pi$ onto the unit circle C, although a 1-1 mapping, is neither open nor closed. A sequence of values of t converging to 2π is closed in our original space whereas its image converges to the point $(1, 0)$ of C and thus is not closed. For 1-1 mappings such as this openness and closedness are equivalent properties and each implies that the mapping is topological. In a general way it may be said that a mapping is open provided the vicinity of a point is not violated or intruded on under the mapping—nothing comes from outside in and approaches the image of the point without actually doing it through the image of the given vicinity. Similarly a mapping is closed provided that a set which wanders off to infinity (i.e., has no limit point) in the original space, has as image a set which behaves similarly—it cannot come in and approach a limit in the image space.

Openness and closedness of a mapping are relative properties in case the image Y of X under f is embedded in a larger space Y_0. In such a situation we say that the mapping is strongly open (or closed) provided the image of each open (closed) set in X is open (closed) in the whole containing space Y_0. This is particularly significant for analytic functions when we usually are considering mappings of regions in the z-plane *into* but not *onto* the w-plane. However, we note that the distinction disappears whenever the image set Y of X is itself open (or closed) in the larger space Y_0.

3. **The Brouwer theorem. A problem.** Perhaps the earliest theorem of note concerned with openness of a mapping was the celebrated one

due to Brouwer to the effect that *Every homeomorphism of a Euclidean space into itself is necessarily strongly open*. This basic proposition seems to have been a powerful influence in the studies of Stoïlow and probably suggested to him the concept of openness for a mapping in general when he had noted that many other mappings, among them the ones generated by analytic functions, have this property. Otherwise stated, the Brouwer theorem asserts that *In a given Euclidean space any set homeomorphic with an open set is itself necessarily an open set*. In this connection it seems worthwhile to suggest consideration of the converse question, namely, *To what extent are the Euclidean manifolds characterized by this property?* In other words, if a suitably restricted space X has the property that each subset of X which is homeomorphic with some open set in X is necessarily open, what, if any, additional conditions are needed to insure that X is a manifold of some dimension? Phrased still another way: *How does one characterize those spaces having the property that any subset homeomorphic with an open set is open?* The answer here is entirely unknown to me, but it probably could be had by some concentrated effort using already available tools in topology and it might well be worth the effort required.

4. **Quasi-compactness.** If instead of requiring that *all* open (or closed) sets in X have open (closed) images, this restriction is limited to open (closed) *inverse* sets, we obtain the notion of a *quasi-compact* mapping. An *inverse* set in X is a set X_0 satisfying $X_0 = f^{-1}f(X_0)$, i.e., a set which is the inverse of its transform under f. Thus a mapping is quasi-compact provided the image of every open inverse set is open— or provided the image of every closed inverse set is closed. Clearly the same concept is obtained whichever way we state the condition because for inverse sets the image of the complement is the complement of the image. Thus quasi-compactness is a weaker condition than either openness or closedness for a mapping and in many respects represents the common ground shared by these two properties. Quasi-compactness is of particular interest in connection with mappings generated by decompositions of a topological space and it was in this connection that the concept was initially formulated by Alexandroff and Hopf [9], who referred to it under the name of *strong continuity*. The natural mapping for any such decomposition of a space X onto the hyperspace X' in which a set is open provided the union of its elements is open in X is always quasi-compact. Further, a given mapping $f(X) = Y$ will be quasi-compact if and only if the decomposition of X into point inverses $f^{-1}(y)$, $y \in Y$, has a

natural mapping $\phi(X) = X'$ which is topologically equivalent to X. Indeed *any mapping $f(X) = Y$ has a topologically unique representation in the form*

$$f(x) = h\phi(x)$$

where ϕ is a quasi-compact mapping and h is a 1-1 mapping. In case f is quasi-compact, h is then a homeomorphism, and f and ϕ are topologically equivalent.

Quasi-compactness is of interest also in connection with the invariance of local connectedness [10]. It has long been known that local connectedness is invariant under all mappings provided the original space X is compact but not in general otherwise. Also it was observed that local connectedness is invariant under open mappings for all types of spaces X. However, it turns out much more generally that this property is invariant under all quasi-compact mappings for all spaces X. Stated in this way this result includes the ones previously mentioned and in addition the invariance of the same property under all closed mappings and under all retractions, because all such mappings are quasi-compact.

5. **Relation to semi-continuity of decompositions.** The exact relationship between quasi-compactness, closedness and openness of a mapping is best exhibited in terms of semi-continuity properties of the decomposition into point inverses generated in the original space by the mapping. In general a decomposition of a space X into a collection G of disjoint closed sets is *upper semi-continuous* (u.s.c.) provided the union of all elements of G intersecting any closed set in X is closed (or, equivalently, the union of all elements containined in any open set is open). Dually, the decomposition is *lower semi-continuous* (l.s.c.) provided the union of all elements of G intersecting an open set in X is open (or, the union of all elements contained in any closed set is closed). We can now state [11] the

THEOREM. *In order that a mapping $f(X) = Y$ be open $\{closed\}$ it is necessary and sufficient that it be quasi-compact and generate an l.s.c. $\{a \ u.s.c.\}$ decomposition of X into point inverses.*

This readily follows. For if f is open or closed it is quasi-compact as already noted. Also if U is any open set in X, $f(U)$ is open if f is open and $f^{-1}f(U)$ is open by continuity and this is exactly the union of all elements of the decomposition intersecting U. Similarly, if f is closed and K is a closed set in X, $f(K)$ and $f^{-1}f(K)$ are closed, giving u.s.c. of the decomposition. The converse follows by an equally simple argument. Thus we see that semi-continuity of the point-

inverse decomposition represents precisely the difference between quasi-compactness and openness or closedness of a mapping.

6. **Quasi-openness.** A word should be added in this connection about another closely related property, namely, quasi-openness. A mapping $f(X) = Y$ is *quasi-open* provided that for any $y \in B$ and any open set U in X which contains a compact component of $f^{-1}(y)$, y is interior to $f(U)$. Similarly f is *strongly quasi-open* if y is interior to $f(U)$ relative to the whole space Y_0 in which Y may be embedded. In case the mapping is light, that is $f^{-1}(y)$ is totally disconnected for each $y \in Y$, quasi-openness is the same as openness but in general not, of course, since openness clearly implies quasi-openness but not conversely. In particular, every monotone mapping on a compact set is quasi-open. This property of quasi-openness is of interest and importance in connection with the topological properties of analytic functions to which we now turn our attention.

7. **Plane region mapping theorem.** Before restricting our attention to the mapping generated by an analytic function we mention an especially useful characterization of quasi-openness for mappings from a plane region into another or the same plane. It reads as follows:

THEOREM. *In order that the mapping $f(X) = Y$ be strongly quasi-open, where X is a plane region and Y lies in a plane π, it is necessary and sufficient that for every elementary region R in X with boundary C in X we have*

$$f(R + C) = f(C) + \text{the union of a set of bounded components of}$$
(*)
$$\pi - f(C).$$

An elementary region is a bounded connected open set whose boundary consists of a finite number of disjoint simple closed curves. To see that quasi-openness implies (*), we suppose some complementary region Q of $f(C)$ in π intersects but does not lie wholly in $f(R)$. Compactness of $f(\overline{R})$ and connectedness of Q then assures the existence of a point $y \in Q \cdot f(R)$ which is a limit point of $Q - Q \cdot f(R)$. Now R contains a component K of $f^{-1}(y)$ and K is compact because $f^{-1}(y)$ cannot intersect C. Then y must be interior to $f(R)$ relative to π by quasi-openness of f, contrary to the fact that y is a limit point of $Q - Q \cdot f(R)$. The converse implication is established by showing first that if U is any open set in the plane P containing X such that U contains a compact component K of $f^{-1}(y)$ for $y \in Y$, then there exists an elementary region R with boundary C such that

$$K \subset R \subset R + C \subset U \quad \text{and} \quad C \cdot f^{-1}(y) = 0.$$

Once this is done, our conclusion readily follows. For y lies in a component Q of $\pi - f(C)$ since it is not in $f(C)$. By (*) we have $y \in Q \subset f(R) \subset f(U)$ so that y is interior to $f(U)$.

8. **Differentiable functions: lightness and openness.** We now consider in some detail the type of mapping from the complex z-plane Z to the w-plane W generated by a function $w = f(z)$ which is nonconstant and differentiable in a region S of Z. It was first noted by Stoïlow that all such mappings are light and strongly open. In other words they are not constant on any continuum, since the set on which the function takes a given value must be totally disconnected (this is lightness) and open sets in S go into open sets in W. Furthermore, these two properties of lightness and openness of all such mappings are the fundamental topological properties of analytic functions in the sense that any topological property of all nonconstant analytic functions necessarily is a consequence of these two. For not only does every nonconstant analytic function have these properties but, conversely, any light open mapping from an orientable triangulable 2-dimensional manifold to the complex plane or sphere is topologically equivalent to an analytic function. This is the primary conclusion on topological properties of analytic functions as originally formulated by Stoïlow.

From the viewpoint of topological analysis the best known method at present for exhibiting lightness and openness for differentiable functions is by means of the circulation index of a mapping about a point, or the winding number about a point. By this method it is possible to prove the properties directly from the assumption of differentiability, making no use of further developments in analysis such as integrals, continuity or zeros of the derivative and the like. This is of importance because many of the results of classical analytic function theory are to be deduced from lightness and openness rather than conversely. The circulation index $\mu_C(f, p)$ of our mapping f about a point $p \in W - f(C)$ taken over a simple closed curve C in the region S of definition of f is defined by taking a mapping $\zeta(x)$ of an interval (a, b) onto C with $\zeta(a) = \zeta(b)$ but which otherwise is 1-1—i.e., a *traversal* of C, and representing the mapping $f\zeta(x) - p$ in the exponential form $e^{u(x)}$ where $u(x)$ is continuous on $(0, 1)$. Then $\mu_C(f, p)$ is defined to be $u(b) - u(a)$. Usually ξ is chosen as a positive traversal of C relative to orientation in the plane Z. For convenience we may define the winding number $\omega_C(f, p)$ of f about p as the number obtained by dividing $\mu_C(f, p)$ by $2\pi i$. Then ω is an integer and measures the net number of times $f(z)$ goes around p when z traverses C once in the positive sense.

Aside from the usual clarifying results on the nature of the circulation index, it can be proven, significantly, that as a function of p, this index is continuous and thus is constant on each complementary domain of $f(C)$ in W and that it vanishes on the unbounded one of these domains. Further, if R is an elementary region lying with its boundary C wholly in S, this index computed over the whole boundary C of R vanishes at a point p of W if and only if p does not belong to $f(R+C)$. In proving this latter statement use is made of the differentiability of f in order to find a point q in the same component of $W-f(C)$ with p such that $f'(z) \neq 0$ for all $z \in f^{-1}(q)$ and to show that for any such z with $f'(z) \neq 0$, $\mu_J(f, q)$ has the value $2\pi i$ for any sufficiently small circle J about z. Once this is established, however, in view of the characterization discussed earlier we then have proven quasi-openness of our mapping f. For if R is any elementary region in S with boundary C in S we must have (*) satisfied because the circulation index vanishes throughout the unbounded component U of $W-f(C)$ so that f takes no value in U; and when f takes on R one value p in a bounded component Q of $W-f(C)$, $\mu_C(f, p) \neq 0$ and hence f takes all other values in Q on R as the index is constant on Q.

Thus we have strong quasi-openness of f; and this means that we will have established strong openness and lightness as soon as we prove lightness. This latter is accomplished by means of the property (*) just noted. Assuming, contrary to lightness, that our function is constant on some continuum in S we are able to construct, with the aid of properties of the circulation index, a function $g(z)$ which will be differentiable inside and on a simple polygon P in S and such that $|g(z)| < \delta = |g(z_0)|/2$ for all z on P where z_0 is inside P. Clearly this contradicts (*). The function $g(z)$ here has the form

$$(\dagger) \qquad g(z) = \prod_{r=0}^{q-1} \{f[(z - z_0)e^{2\pi i r p/q} + z_0] - a\}.$$

For details of this argument as well as other proofs mentioned in this connection reference is made to the previously mentioned summaries of mine together with articles referred to therein—in particular here to a paper of Ursell and Eggleston [7] which contains some of the basic ideas which make the proofs successful.

9. **Local and global analysis of light open mappings.** The important converse conclusion: that every light (strongly) open mapping from a 2-manifold to the complex plane or sphere is topologically equivalent to an analytic function is best treated in the more general setting of light open mappings acting on 2-manifolds. We only out-

line the treatment in brief since it has been discussed in detail in the sources quoted earlier. In the first place it can be shown that the property of being a 2-manifold is invariant under light open mappings. Thus it is not necessary to assume such a highly simplified structure of the image space. In the next place, although we assume only that the inverse of each point is totally disconnected, it follows from results of plane topology closely related to the Jordan Curve Theorem that each point inverse is a completely scattered set, that is, if our mapping f is constant on a set X, then no point of X is a limit point of X. Thus the situation is greatly simplified and we are in position to complete a local analysis of the action of the mapping. This consists in showing that if $f(A) = B$ is our mapping where for simplicity we suppose A and B are 2-manifolds without edges, then for any $y \in B$ and any $x \in f^{-1}(y)$ there exists 2-cell neighborhoods V of y and U of x such that $f(U) = V$ and the mapping of U onto V is topologically equivalent to a power mapping $w = z^k$ on $|z| \leq 1$ for some integer k.

Since it also follows in this situation that orientability of B implies that of A, it is now possible, in case B is a region on the complex plane or sphere, to construct a Riemann surface Σ onto which A can be mapped by a homeomorphism $h(A) = \Sigma$; and, using standard procedures of analysis, an analytic function $\phi(z)$ can be constructed on Σ in such a way that the relation

$$f(x) = \phi h(x)$$

is satisfied for all $x \in A$. Thus f is topologically equivalent to the analytic function ϕ.

10. Maximum modulus results. Rouché's Theorem. Degree and zeros. Among the numerous well known fundamental results of analytic function theory which are direct consequences of lightness and openness of the associated mapping brief mention will be made of only a few. These include the Maximum Modulus Theorem in its full strength together with a group of related results on existence of zeros such as the Fundamental Theorem of Algebra. Also, Rouché's Theorem in its full generality is readily obtainable using easily developed properties of the circulation index, as are also other classical results concerning the number of zeros and poles of a meromorphic function inside a simple closed curve on which it is analytic and $\neq 0$. Homotopy classification of analytic and meromorphic functions can be clarified by similar methods and greatly illuminates the topological character and action of the mappings generated by such functions.

Studies along these latter lines have been carried out in recent years by Kuratowski with striking success.

11. Sequence results. In connection with sequences of functions and of mappings, two theorems fit together in a peculiarly interesting way. They read as follows:

1. *If A and B are locally compact connected and locally connected separable metric spaces, and if $f_n:A \to B$, $n = 1, 2, \cdots$, is any sequence of strongly quasi-open mappings of A into B which converges uniformly to the mapping $f:A \to B$, then f is strongly quasi-open.*

2. *If the sequence of functions $w_n = f_n(z)$, each differentiable in a region S of Z, converges uniformly in S to a nonconstant function $w = f(z)$, then f is light.*

Thus by No. 1 quasi-openness carries over to the limit function from members of the sequence. In general lightness does not carry over for sequences of mappings—indeed not even for sequences of homeomorphisms. Remarkably, however, by No. 2 it does carry over for uniformly convergent sequences of differentiable functions. Now applying No. 1 to the case covered by No. 2 we see at once that, in No. 2, f is also strongly quasi-open because this holds for each f_n. Thus, in No. 2, f is both light and strongly open and hence has the characteristic topological character of an analytic function. We have then a theorem giving the topological content of the Weierstrass Double Series Theorem.

Now using the lightness of f as given in No. 2, which enables us to apply the circulation index to f, the theorem of Hurwitz is readily obtained. This theorem asserts that, under conditions in No. 2, if ζ is an m-fold zero of $f(z)$, then every sufficiently small neighborhood D of ζ contains exactly m zeros of $f_n(z)$ for $n > N(D)$. This important conclusion is thus obtained purely from the topology of the situation with no need for knowledge of the differentiability of the limit function f.

12. Dimension and nondensity preservation of mappings. General setting. The remainder of this lecture will be devoted to a consideration of results and questions concerned with dimension and nondensity preservation of mappings and their applications to differentiable functions. The results to be mentioned are largely new ones, announced here for the first time. We begin by recalling a simple example of a monotone mapping of a square S onto a square Σ which alters dimension of a compact nondense subset. This example is not new, though perhaps it is put to a new use. This mapping can be effected by dividing S into 9 equal squares and mapping the middle one onto

the center of Σ; then divide each of the 8 remaining squares into 9 equal squares and map the 8 middle ones onto 8 points symmetrically distributed in Σ and so on. This procedure can be continued so as to obtain a uniformly continuous mapping of a set of squares in S with union dense in S onto a set of points dense in Σ; and the continuous extension of this mapping is our required mapping ϕ of S onto Σ. When carefully described, ϕ will be monotone and if K denotes the complement in S of the union of the interiors of the center squares selected in the definition, then K is nondense and thus 1-dimensional whereas $\phi(K) = \Sigma$ and thus is 2-dimensional. Also, it should be noted that on the set $D_\phi = S - U$, where U is the union of all center squares, ϕ is 1-1 and D_ϕ is dense in K and $\phi(D_\phi)$ is dense in Σ. However, D_ϕ is not dense in any open set in S.

Thus it is possible to alter dimension of a compact subset under a mapping of a quite restricted type even when the dimension of the whole set is not changed. This fact is not new by any means as I believe it was mentioned by Menger [12] in his early papers on dimension theory in connection with this same example. In this connection we recall next a theorem of Alexandroff's [13] to the effect that under open mappings having totally imperfect point inverses, dimensionality of the whole space is not altered provided the original space X is separable, metric and locally compact and the image space is metric. This result does not apply to show that the dimension of subsets of X are not altered under these conditions, because the openness assumed on X does not necessarily hold on the subset of X in question. However, using the method of Alexandroff, it is readily shown that *if X and Y are locally compact, separable and metric and $f(X) = Y$ is an open mapping having scattered point inverses, then X is the union of a countable sequence of compact sets X_n such that $f|X_n$ is topological*. It follows from this, of course, that for any closed set $K \subset X$, dim $f(K) = $ dim K. This result in this form is of importance and will be used later in connection with our discussion of differentiable functions.

Now for a mapping in general, $f(X) = Y$, we let L_f denote the set of all $x \in X$ such that $f^{-1}f(x)$ is totally disconnected and D_f the set of all $x \in X$ such that x is a component of $f^{-1}f(x)$. Obviously $L_f \subset D_f$ for any f. We shall say that f *preserves nondensity* for compact sets provided that if K is a compact nondense set in X, i.e., K contains no open set in X, then $f(K)$ is nondense in Y. If X and Y are locally compact separable and metric and the mapping $f(X) = Y$ preserves nondensity of compact sets, it turns out that $L_f \neq 0$. Indeed it can be proven that $f(L_f)$ *is dense in B and L_f itself is semi-dense X*, i.e.,

dense in some open subset of every open set U in X with an open image. The proof here presents little difficulty and follows readily using countable coverings by small open sets and taking unions of boundaries. It may be remarked also that the same conclusion about L_f can be obtained under the assumption that f preserves nondensity for compact sets K with dim $K < k = $ dim $X < \infty$. It is of interest to note again that in the example of the mapping ϕ just given, $f(L_\phi)$ is dense in Y but L_ϕ is not dense in any open set whatever of S. Here L_ϕ and D_ϕ are the same, of course, since ϕ is monotone.

13. **Quasi-open mappings on 2-manifolds.** We now limit ourselves to the specific setting in which the original space X is a 2-manifold and where the mapping is quasi-open. We continue using the notion of *semi-density* in the sense defined in §12. First we have the

THEOREM. *If X is a region on a sphere, $f(X) = Y$ is compact and quasi-open and no component of a point inverse separates X, then dim $f(K) \leq $ dim K for all compact 1-dimensional sets K in X if and only if D_f is semi-dense in X.*

Compactness of the mapping f here means that the inverse of every compact set is compact or, equivalently, that f is closed and point inverses are compact. The "only if" part of this theorem is an immediate consequence of the result just quoted for mappings in general, because $D_f \supset L_f$. The proof of the reverse implication, that density of D_f is an open subset of every open set in X with an open image implies the asserted dimension invariance is effected by an interesting combination of factorization of the mapping along with the invariance of dimension under open mappings with scattered point inverses quoted earlier. For compactness of f enables us to factor f into monotone and light factors:

$$f = lm, \qquad m(X) = X', \qquad l(X') = Y.$$

Then the monotone mapping m can be "extended" to the whole sphere S on which X lies by decomposing S into the sets $m^{-1}(x')$, $x' \in X'$, together with components of $S - X$. The natural mapping $\phi(S) = S'$ of this decomposition maps S monotonically onto another sphere S' by a classic result of R. L. Moore [14]. Then $l(X') = Y$ is a light, open, compact mapping of X', a region on S', onto a set Y which is necessarily a 2-manifold. Thus l necessarily has finite point inverses. Our density assumption for D_f insures that dim $m(K) \leq 1$ when $K \subset X$ is compact and of dimension ≤ 1 because m is 1-1 on D_f, of course, and dim $lm(K) \leq 1$ since l is open and has finite point inverses.

The general case of a quasi-open mapping $f(X) = Y$ of an arbitrary 2-manifold X without edge points onto a locally connected generalized continuum Y can be reduced essentially to the case just discussed where X is a region on a sphere by means of the following:

LEMMA. *Suppose $f(L_f)$ is dense in Y and that for some compact nondense set K in X, $f(K)$ contains an open set in Y. Then there exists a region R in X contained in a 2-cell of X such that $Q = f(R)$ is open in Y, the mapping $f(R) = Q$ is compact and quasi-open and for some compact subset K_1 of $K \cdot R$, $f(K_1)$ contains an open set.*

This reduction is effected by taking a point $y \in f(L_f) \cdot V$ where V is an open set in $f(K)$ and covering each $x \in K f^{-1}(y)$ by a 2-cell in X whose edge does not intersect $f^{-1}(y)$. After reducing this to a finite covering, it can then be shown that one of these 2-cells includes a region R meeting our conditions. Using this reduction we can then apply the previous theorem and thus handle the general case embodied in the

THEOREM. *Given a quasi-open mapping $f(X) = Y$ of a 2-manifold X without edge points onto a locally connected generalized continuum Y such that no component of a point inverse lying inside a closed 2-cell on X separates X. In order that f preserve nondensity for compact sets it is necessary and sufficient that the set L_f be semi-dense in X.*

In case the space Y is a 2-manifold, or if it has the property that every subset of dimension ≥ 2 contains an open set, the conclusion of *preserving nondensity* clearly is the same as *preserving the property of being of dimension ≤ 1* for compact subsets. If $f^{-1}(y)$ has a degenerate component (or a component lying in a 2-cell of X which it does not separate) for every $y \in Y$, then Y will be a 2-manifold and the theorem could then be stated in terms of dimension preserving.

14. **Differentiability and dimension raising.** Returning now to the case of a function $w = f(z)$ continuous on a region X of Z we note at once that if $f(z)$ is differentiable everywhere in X, then the mapping is light and open so that $L_f \equiv X$. Accordingly by the theorems just discussed, any compact nondense set K in X has a nondense image set or, in other words, $\dim f(K) \leq \dim K$. This fact also is a consequence of the theorem stated earlier about open mappings with scattered point inverses, because in this situation f not only is light but has scattered point inverses. However, it is of interest and of some importance to know to what extent the differentiability assumption can be weakened. Making no use of the lightness and openness of differentiable functions it is still possible to prove the following

THEOREM. *Let $w = f(z)$ be continuous in a region X of Z and differentiable at all points of $f^{-1}(Y_0)$ for some open subset Y_0 of Y dense in Y. Then f is strongly quasi-open, no component of a point inverse lying inside a closed 2-cell in X separates X and the set $f(L_f)$ is dense in Y. Further, if f is not constant on any open set in X, L_f is semi-dense in X.*

Under the condition last stated, all conditions of the theorem for 2-manifolds are satisfied so that f preserves nondensity for compact sets, or does not raise dimensionality for any compact set of dimension ≤ 1. In this connection it is of interest to note that in the example given earlier of a monotone mapping from a square S to a square Σ, the mapping function ϕ there is differentiable and $\phi' \equiv 0$ in the open everywhere dense subset U of S consisting of the union of all interiors of center squares selected. However, U is not an inverse set nor is its image open in Σ. As already noted here, although $f(L_\phi)$ is dense in Σ, L_ϕ itself is not dense in any open set whatever of S.

15. Applications. Concept of a pole. We now apply these results to a situation in which dimension or nondensity preservation is important, as are also reduced differentiability assumptions. Suppose our function $w = f(z)$ is nonconstant and continuous in a region R of Z and differentiable on $R - H$ where H is a closed nondense subset of R on which f is constant. Can it still be asserted that f is light so that in particular H is totally disconnected? The answer here is: Yes, it can. However, the proof is far from easy and it involves these same issues we have been discussing. For in the proof of lightness of a differentiable function referred to earlier, the new function g which was mentioned is defined as a product of a finite number of factors each of which is a value of f at a point obtained by rotating the given z about a fixed center z_0 [see (†) above]. Thus we can be sure of differentiability of g only on the complement of the union of the images of H under the finite number of rotations involved. However, it can be shown that g will be differentiable on everywhere dense open inverse sets and hence our previous theorem applies to give us strong quasi-openness of g and this is exactly what is needed to show that H is totally disconnected. This yields the following results, listed A − D.

A. *If $w = f(z)$ is nonconstant and continuous in a region R and differentiable on $R - f^{-1}(a)$ where a is some value of f in R, then f is light and strongly open in R.*

It could be asserted, further, that f has completely scattered point inverses, so that $f^{-1}(a)$, in particular, is completely scattered. Also f is locally topologically equivalent to a power mapping even at points of $f^{-1}(a)$.

B. *Let $w = f(z)$ be differentiable in $R - K$ where K is a closed non-dense subset of R such that for each $z_0 \in K$ we have $\lim_{z \to z_0} f(z) = \infty$. Then K is a completely scattered set and the mapping of R into the complex sphere is light and strongly open.*

Thus all points of K are topologically precisely like poles of f. Hence the concept of a pole is approachable topologically. No reference to expansions in series need be made. A pole is simply a point where $f(x) \to \infty$ and such that in some neighborhood of this point there are only points of this same type and points where f is finite and differentiable.

C. *Let $w = f(z)$ be nonconstant and differentiable in a region R. Then if f is continuous at all points of a continuum K of $\mathrm{Fr}(R)$ and constant on K, K is a continuum of condensation of $\mathrm{Fr}(R)$.*

D. *In particular, in C, if R is an elementary region, there can exist no arc in $\mathrm{Fr}(R)$ on which f is continuous and constant. Thus if f is continuous on $\mathrm{Fr}(R)$, it must be light on $\mathrm{Fr}(R)$.*

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