DOMAINS OF POSITIVITY

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A Domain of Positivity D is an open convex cone associated with a nonsingular symmetric matrix S, called the characteristic, such that $x \in D$ if and only if $x'Sy > 0$ for all $y \in \bar{D}$. As such they were introduced by Koecher (1) in generalization of the cone of positive definite matrices studied by Siegel. The automorphisms of D are the nonsingular linear transformations mapping D onto itself. The group of automorphisms $\{W\}$ admits an anti-automorphism: $W \rightarrow S^{-1}W'S$, where W' means W transposed. A norm $N(x)$ is a function positive and continuous for $x \in D$ and satisfying there $N(Wx) = \|W\| N(x)$ for every automorphism W. A norm is given by:

$$1/N(x) = \int_D \exp(-x'St)dt$$

and a group invariant positive definite metric form is given by:

$$g_{ij} = -\frac{\partial^2 \log N(x)}{\partial x_i \partial x_j} \cdot$$

The Domain is called homogeneous if the automorphisms are transitive. In this case the Domain has an involution given by:

$$x \rightarrow x^* = S^{-1} \text{ grad } \log N(x).$$

For homogeneous domains it is easy to show that $N^2(x)$ is always a rational function. If the characteristic is positive definite much more is true. In the first instance, the fixed points of the anti-automorphism of the group of automorphisms already act transitively on the domain D. It follows that the norm satisfies the important equality:

$$N(x^* + y^*) \cdot N(x) \cdot N(y) = N(x + y).$$

Moreover, for every point x in D there is an involution of the Domain keeping x fixed. Hence D is a symmetric (Cartan) space, and it is possible to make a detailed study of the Lie group of automorphisms. The following facts emerge:

(a) $N^2(x)$ is a polynomial,

(b) The geodesic connecting any two points (given by Cartan's construction of geodesics in a symmetric space) is unique,

(c) The Domain has everywhere zero or negative curvature.

(d) The involution on D extends to an analytic involution of the whole of the tube with D as base onto itself.

On the basis of (a) it follows from a result of Bochner (2) that the so-called "Gamma-Factor" of the Domain is indeed a product of Gamma functions. From (d) together with the equality described above satisfied by the norm, it follows from another result of Bochner (3) that there exist unitary transformations of $L_2(D)$ relative to the volume element $N^s(x)$ for suitable values of s, which generalize the Hankel Transform.

In quite another direction is the result, true for any homogeneous domain that, except for a multiplicative constant, $1/N^2(z+\bar{w})$ is the Bergmann reproducing kernel for analytic L_2 in the tube with D as base. This result shows quite clearly that the involution on D cannot extend to a holomorphic mapping of the tube with D as base unless the norm satisfies the equality stated earlier.

We understand Koecher has proved some of the same results, though they are not yet published at this writing.

References

1. Max Koecher, *Positivitatsbereiche im R^n*, Amer. J. Math. vol. 79, no. 3, July, 1957.

2. S. Bochner, *Gamma factors in functional equations*, Proc. Nat. Acad. Sci. vol. 42, no. 2, February, 1956.

3. ———, *Bessel functions and modular relations of higher type and hyperbolic differential equations*, Communications du Seminaire Mathematique de l'Universite de Lund, tome supplementaire, 1952, dedie a Marcel Riesz.

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