solution of ordinary differential equations, simultaneous linear equations, and elliptic partial differential equations via partial difference equations. These topics are covered adequately if not liberally as far as an introductory course would be concerned. Two final chapters on parabolic and hyperbolic equations and integral equations are perfunctory.

The text is liberally strewn with worked examples, and includes many problems, some theoretical, and some requiring numerical work with desk machines. Round-off, truncation, and instability, the three devils of numerical analysis, are introduced in such a way as to drive the student from the paradise of infinitely precise computation, without plunging him into the hell of infinitely precise error analysis. There is no direct treatment of electronic computers as such, but here and there are interpolated remarks to the effect that such and such technique is or is not well suited to automatic computers. There are a good many references to the recent literature on numerical analysis, and this way, the student should get the feeling that numerical analysis did not come to an end with Horner. On the other hand, these references are spotty; many references that one would like to see are not present, and this reviewer is left with the impression that the author was not clear how much to update his original notes of 1947–1949 in the face of the recent flux in the field.

PHILIP J. DAVIS

Primzahlverteilung. By Karl Prachar. Berlin, Göttingen, Heidelberg, Springer-Verlag, 1957. 10+415 pp. DM 55. Bound DM 58.

This treatise is a distinguished sequel to Landau's monumental *Handbuch*. It contains a skillfully presented, up-to-date and extensive account of that recondite branch of number theory—the analytic theory of the distribution of primes. Within its ten chapters are incorporated many remarkable new results never previously treated in any book. Above all, here may be found: Selberg's improvement of the Viggo Brun sieve technique; various interesting results of Erdös on the difference of consecutive primes; the theorem that almost every even integer is representable as the sum of two odd primes; Tatuzawa's proof of Rodosskii's theorem on the distribution of primes in "short" arithmetic progressions; the best known error term in the prime number formula; the theorems of Hoheisel, Ingham and Tatuzawa on the difference of consecutive primes; Rodosskii's proof of the celebrated theorem of Linnik on the smallest prime in an arithmetic progression.

Because of the enormity of his program the author reluctantly abandons the ideal of absolute completeness. For example, he does not consider the so-called "large sieve" of Linnik and Rényi's refinement of it. He is thus compelled to omit Rényi's theorem that every large even integer can be written as the sum of a prime and an "almost" prime. Again, he does not develop the intricate apparatus required to adopt the Brun sieve to obtain estimates from below.

Despite its omissions the book marshals an imposing array of ideas and theories. Prominently featured are the researches of Vinogradoff and his school, which have transformed much of analytic number theory since the year 1934. The uncanny ingenuity, depth and power of the new methods continue to astonish us almost a quarter of a century after their inception!

It is now fifty years since the analytic theory of numbers was first presented as a systematic science by Edmund Landau in the two volumes of his *Handbuch der Lehre von der Verteilung der Primzahlen.* Its coverage of every phase of the subject as it was known at that time was all embracing. The results of later researches were amalgamated by Landau in his 1927 *Vorlesungen über Zahlentheorie.* In a joint tribute to Landau [J. London Math. Soc. vol. 13 (1938) pp. 302–310] Hardy and Heilbronn commented that these pioneering books "transformed the subject, hitherto the hunting ground of a few adventurous heroes, into one of the most fruitful fields of research of the last thirty years." The next twenty years have seen a continued upsurge of interest in analytic number theory, and several important books on the subject have appeared. We may mention, for example, the Cambridge Tracts of Ingham and Estermann, the monograph of Trost, Titchmarsh's book on the Riemann zeta function, Vinogradoff's book on trigonometric sums, the book of Hua on additive prime number theory, and the book of Tschudakoff on Dirichlet's L functions. But all of these later works, outstanding as they are, are restricted in scope. The literature of our subject has indeed been enriched with the appearance of the book under review, and it is especially welcome.

Wisely the author does not caricature either variation of the characteristic and inimitable Landau style. He eschews the majestic sweep of the *Handbuch* and the austere exactitude of the *Vorlesungen.* Nevertheless his presentation is masterful and scholarly in its own right. He starts from the beginning, establishes his prospectus, and synthesizes his material. The result is a fine book which, in spite of the great difficulty and complexity of the subject matter, is self-contained, balanced, thorough, meticulous and pleasantly written.

The contents of the book, with its wealth of theorems, can only be summarily indicated in this review. The book opens with a succinct historical exordium. Chapter One is concerned with preliminaries. The Tschebyscheff inequalities for $\pi(x)$, the number of primes not exceeding a real number x, are derived; estimates are obtained for various sums and products involving primes; and applications to the totient function and divisor function are given.

The author gives an illuminating account of Selberg's sieve method in Chapter Two. By removing certain restrictions inherent in the older Viggo Brun technique Selberg has developed the sieve into an extremely versatile and flexible instrument. In spite of its natural limitations the capabilities of the method have not been fully explored. In this book only upper bounds are derived. Among the many applications given by the author is Brun's striking theorem: if p' runs over the twin primes, then the series $\sum 1/p'$ is at most convergent. But the age old conjecture that there are an infinitude of twin primes remains a major challenge to the ingenuity of mathematicians! Goldbach's problem may also be attacked by Selberg's sieve. For instance, Selberg has proved that every large even integer can be written as the sum of two positive integers, one of which contains at most two and the other at most three prime factors. It is regrettable that the author only alludes to Rényi's method for establishing lower bounds. Using his method Rényi in 1948 proved that there are infinitely many primes differing from "almost" primes by two. By an "almost" prime is meant a number containing no more than a fixed (but unspecified) number of prime factors.

The prime number theorem states that $\pi(x) \sim x/\log x$ and was first proved by Hadamard and de la Vallée Poussin in 1896. De la Vallée Poussin proved furthermore the refinement

$$\pi(x) = \int_2^x \frac{dt}{\log t} + R(x)$$

for sufficiently large x, with $|R(x)| < c_1 x e^{-c_2(\log x)^{1/2}}$, c_1 and c_2 being absolute positive constants. The proofs of both Hadamard and de la Vallée Poussin depend upon Hadamard's theory of entire functions, and in particular on the fact that $\zeta(s)$, apart from a simple pole at $s=1$, is regular all over the complex plane. In Chapter Three a simplified proof due to Landau is presented. The Hadamard theory, the Weierstrass product and the functional equation are not needed. Indeed the essential ingredient of the proof is the knowledge of a certain zero-free region of $\zeta(s)$ slightly to the left of the line $R(s)=1$. The author proceeds to the development of a body of theorems related to and logically interdependent with the prime number theorem.

Notable among these is the famous identity (Euler, von Mangoldt) of Landau's doctoral dissertation $\sum_1^\infty \mu(n)/n=0$. Recent discoveries have emphasized that the classification of a theorem of prime number theory according to depth is relative to the existing state of knowledge. Chapter Three concludes with the now almost classical elementary proof of the prime number theorem based upon Selberg's fundamental asymptotic formula. In this connection the author adroitly gives star billing to both Erdös and Selberg. To establish Selberg's formula the author follows the simplified proof of Iseki and Tatuzawa; he apparently has overlooked the fact that essentially the same proof was first published by Shapiro [Ann. of Math. (2) vol. 50 (1949) pp. 305–313]. The passage to the prime number theorem is then achieved via the method of R. Breusch.

It was to prove his ground-breaking theorem on primes in an arithmetic progression that Dirichlet in 1837 created his brilliant theory of characters. Let $k\geqq 1$ and $0\leqq l<k$, $(l, k)=1$. Then there exist infinitely many primes p such that $p\equiv l \pmod{k}$. The author, in Chapter Four, gives Dirichlet's proof as simplified by Mertens and Landau. He also sketches the elementary proof of Shapiro (1950). Let now $\pi(x, k, l)$ denote the number of primes p satisfying $p\leqq x$ and $p\equiv l \pmod{k}$. Employing the theory of the Dirichlet $L(s, \chi)$ functions de la Vallée Poussin proved the prime number theorem for arithmetic progressions. Speaking crudely we may say that he proved that in each of the $\phi(k)$ relatively prime residue classes modulo k there are "equally many" primes. More precisely he proved that

$$\pi(x, k, l) = \frac{1}{\phi(k)}\int_2^x \frac{dt}{\log t} + R(x, k, l),$$

with $|R(x, k, l)| < c_1 x e^{-c_2(\log x)^{1/2}}$, where $c_1=c_1(k)$, $c_2=c_2(k)$ are positive constants. For the proof of this theorem it is necessary to determine the behavior of $L(s, \chi)$ in the strip $0\leqq R(s)\leqq 1$; the zeros of $L(s, \chi)$ play a role analogous to those of $\zeta(s)$ in the study of $\pi(x)$. Note that the estimate of the error term $R(x, k, l)$ depends on k as well as x. In a series of researches initiated by Landau, Titchmarsh and Page the effect of the delicate interplay between k and x on the estimation of $R(x, k, l)$ has been investigated. There is, for example, an important theorem of Page which asserts that the above estimate of $R(x, k, l)$ is uniform in k for $k\leqq e^{c(\log x)^{1/2}}$ and x large unless k belongs to a certain set of exceptional values. Page's theorem has turned out to be an indispensable tool for handling many difficult problems of additive number theory [e.g. G. L. Watson's proof that every sufficiently large integer is the sum of seven positive cubes].

Its demonstration depends on the theory of L series as developed by Dirichlet, Riemann, Hadamard, de la Vallée Poussin, Hardy and Littlewood, Landau and others before the year 1935. The theorem of Page has subsequently been refined by Walfisz (1936). The proof of the theorem of Walfisz is based on Siegel's lower estimate of the value at $s=1$ of the $L(s, \chi)$ functions with real characters χ. The author gives Estermann's surprisingly simple proof of Siegel's result.

Sundry applications of the results in Chapters Two and Four are presented in Chapter Five. First, the sieve method is applied to the problem of obtaining bounds for $\pi(x, k, l)$. A typical result is the theorem of Titchmarsh. Let $0<a<1$ and $1 \leqq k \leqq x^a$. Then $\pi(x, k, l) < cx/\phi(k) \log x$, $c=c(a)$ for all l, $(l, k)=1$, $0 \leqq l<k$. An exact analogue of this inequality for lower bounds is not known. Density theorems are next studied. The theorem of Schnirelman that the integers representable as the sum of two odd primes have positive asymptotic density is proved. So is the theorem of Romanoff on the representation of integers in the form $p+a^m$. The theorems of Titchmarsh and Erdös on the number of divisors of $p-1$ are also derived. Finally, there is given a comprehensive treatment of the theory of the difference of consecutive primes. Included are theorems associated with the names of Knödel, Prachar, Erdös, Rankin, Chang, De Bruijn, Sierpinski and Walfisz.

The conjecture of Goldbach, enunciated by him in 1742, that every even integer greater than four is the sum of two odd primes, awaits proof or disproof. For almost two centuries this most difficult problem of additive number theory was intractable. In 1922 Hardy and Littlewood introduced the powerful Farey dissection method into analysis and proved on the basis of an unproved conjecture about the Riemann zeta function that every sufficiently large odd number can be represented as the sum of three odd primes, and that "almost" every even number is the sum of two primes. In 1937 Vinogradoff proved, in a sensational paper, the first of the Hardy-Littlewood theorems without employing any hypothesis. Later van der Corput, Estermann and Tschudakoff gave complete and independent proofs of the second theorem. Chapter Six is devoted to an exposition of Vinogradoff's method. It should be pointed out that Vinogradoff's result is in the nature of an existence theorem; it does not reveal explicitly a point beyond which all odd numbers are sums of three primes. It is therefore comforting to the reviewer to know that Rosser and Schoenfeld have proved (but have not yet published) the theorem that every odd integer with at least 350,000 digits is the sum of three odd primes.

In the long seventh chapter the author delves deeper into the function theoretic properties of the L functions. The successive section headings are: the functional equation; the partial fraction decomposition of $L'(s, \chi)/L(s, \chi)$; further theory of the distribution of zeros of $L(s, \chi)$; "explicit formulae" theorems; consequences of the Riemann hypothesis; another "explicit formula"; on the smallest prime in an arithmetic progression; irregularities of prime number distribution. The "explicit formula" theorems exhibit a curious type of connection, by exact infinite series representations, between the sums $\psi(x, \chi) = \sum_{n=1}^{x} \chi(n)\Lambda(n)$ and the zeros $\rho = \rho(\chi)$ of $L(s, \chi)$ in the critical strip. The principal results in the penultimate section are the least prime theorems of Chowla and Turan. A major theorem of the final section is the now classical result of Littlewood that $\pi(x) > li\ x$ infinitely often, where $li\ x$ is the logarithm-integral. Littlewood did not make any estimate of the smallest value of x for which the inequality is true. This problem has been studied by Littlewood's student Skewes over a period of many years. The latest (1955) result of Skewes is that the number does not exceed exp exp exp exp (7.705). Much earlier Hardy had already remarked, "However much the number may be reduced by refinements on Skewes's argument, it does not seem at all likely that we shall ever know a single instance of the truth of Littlewood's theorem."

Until 1921 the error term in de la Vallée Poussin's prime number theorem was the best known result of its type. In that year Littlewood employed the theory of Weyl sums to prove that $|R(x)| < c_1 x e^{-c_2(\log x \log \log x)^{1/2}}$. In 1935, Tschudakoff used Vinogradoff's abstruse method for estimating trigonometric sums to derive $|R(x)| < c_1 x e^{-c_2(\log x)^a}$, for a value of a between 1/2 and 1, and x large. We are indebted to Vinogradoff and Hua for the best result to date (1951):

$$| R(x) | < c_1 x e^{-\lambda(x)}, \qquad \lambda(x) = c_2 \frac{\log^{4/7} x}{(\log \log x)^{3/7}} .$$

The proof given in Chapter Eight is based upon ideas of Tatuzawa. The crux of the argument is that the best known result concerning the region left of $\sigma = 1$ (with large t) in which $\zeta(s) \neq 0$ leads to the prime number theorem with a corresponding best known error term. To obtain such a zero-free region the method of Vinogradoff for dealing with the exponential sums arising in the approximate formulae for $\zeta(s, w)$ (Hurwitz zeta function) is used. Actually, the deeper theory of the zeta function is not required.

Tschebyscheff gave the first proof of Bertrand's postulate: if p_n

denotes the nth prime, then $p_{n+1}<2p_n$. Legendre conjectured, but no one has ever proved, that $p_{n+1}-p_n<p_n^{1/2}$ for all sufficiently large n. Hoheisel in 1930 established the existence of a number a, $1-1/33000<a<1$, such that $p_{n+1}-p_n<p_n^a$. The exponent a was successively diminished by Heilbronn in 1933, by Tschudakoff in 1936 and by Ingham in 1937. Ingham obtained $a=5/8$ and also a somewhat smaller value. The proofs of these and related results make use of the theory of the density of the zeros of the zeta function. In Chapter Nine the machinery of this theory is developed. Applications are given also to the work of Linnik (1943, 1945), Rodosskii (1949), Tatuzawa (1950) and Haselgrove (1951) on the distribution of primes in "short" arithmetic progressions. Further applications concern the estimation of $\zeta(1/2+it, w)$.

The crowning achievement of the last chapter is the deep theorem of Linnik: Let $k\geqq 2$, $(l, k)=1$, $l<k$, and let $p_1(k, l)$ be the smallest prime in the arithmetic progression $nk+l$, $n=1, 2, \cdots$. Then there exists a constant C independent of k such that $p_1(k, l)<k^C$. The awe-inspiring proof involves forty pages and twenty-one lemmas.

The book closes with an Appendix. This contains a brief summary of pertinent theorems and formulae from the theory of functions.

The author is to be congratulated for having written an important and valuable book. The House of Springer is to be congratulated on a superb example of the art of mathematical printing.

ALBERT LEON WHITEMAN

Neue topologische Methoden in der algebraischen Geometrie. By F. Hirzebruch. Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, vol. 9. Springer, 1956. 165 pp. DM 30.80.

This book, devoted to the topological transcendental theory of algebraic varieties over the complex field, should rank with Lefschetz's *L'Analysis situs et la géométrie algébrique*, Paris, 1924, and Hodge's *Harmonic integrals*, Cambridge, 1941, as a milestone in the development of the theory. While topology plays the essential rôle in Lefschetz's book and Hodge's main tool is harmonic differential forms, this book is characterized by the diversity of deep and difficult results which the author drew for his use. These include, among others, Todd's genus, Thom's algebra, and Kodaira's work on complex manifolds. Sheaves (or stacks or faisceau in French and Garbe in German) and analytic bundles with their characteristic classes are the pillars on which the main result is built.

The main result, which is not proved until the very end of the book, is the Riemann-Roch Theorem for nonsingular complex alge-