ABSTRACT CAUCHY PROBLEMS OF THE ELLIPTIC TYPE

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Let A be the infinitesimal generator of a strongly continuous one-parameter semigroup $T(\xi)$, $0<\xi$, of endomorphisms over a B-space X. Suppose it is required to find a function $u(t)$, $0<t$, with values in X such that:

(i) $u(t)$, $u^1(t), \cdots, u^{n-1}(t)$ are absolutely continuous, $u^k(t)$ being the derivative of $u^{k-1}(t)$.

(ii) $u^n(t)=(-1)^{n+1}Au(t)$.

(iii) $\|u^k(t)-u_k\|\to 0$, as $t\to 0+$, $k=0,\cdots,n-1$.

We call this an abstract Cauchy problem of the elliptic type (ACPE_n). We prove:

THEOREM 1. *The $ACPE_n$ has at most one solution provided*

$$(\mathrm{H}_1) \qquad \int_1^\infty \|T(\xi)\|\xi^{-\sigma-1}d\xi < \infty \text{ for every } \sigma > 0.$$

THEOREM 2. *Let $n=2$. Let the semi-group $T(\xi)$ satisfy* H_1 *and let $u(t)$ be any solution of the $ACPE_2$ such that*

$$(\mathrm{H}_2) \qquad \limsup_{t\to\infty} t^{-1}\,\mathrm{Log}\,\|u(t)\| \leqq 0.$$

Then necessarily

$$(1) \qquad u(t) = (t/2\pi^{1/2})\int_0^\infty T(\xi)u_0\xi^{-3/2}\exp(-t^2/4\xi)d\xi.$$

A slightly different but useful version of Theorem 2 is:

THEOREM 3. *Let $n=2$. Let the semi-group $T(\xi)$ satisfy* H_1. *Let $u(t)$, $t>0$, satisfy* (i), (ii), *but* (iiia) *below in place of* (iii)

$$(\text{iiia}) \qquad \|u(t)-u_0\| \to 0 \text{ as } t\to 0+.$$

Then, if $u(t)$ satisfies H_2 *in addition, $u(t)$ is again determined by* (1). *Moreover, if $\|T(\xi)\|\to 0$ as $\xi\to\infty$, then any such $u(t)$ has a similar property, viz.:*

$$\|u(t)\| \to 0, \text{ as } t\to\infty.$$

Results similar in principle have been obtained for other values of n.

As an application of these results we may consider the elliptic equation:

$$\frac{\partial^2}{\partial t^2} u(t, x) + a(x) \frac{\partial^2}{\partial x^2} u(t, x) + b(x) \frac{\partial}{\partial x} u(t, x) = 0, \tag{2}$$

where the functions $u(t, \cdot)$ are to lie in $C[\alpha, \beta]$, $-\infty \leqq \alpha < \beta \leqq \infty$, $a(x)$, $b(x)$ are continuous and $a(x) > 0$. It suffices to consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} u(t, x) = a(x) \frac{\partial^2}{\partial x^2} u(t, x) + b(x) \frac{\partial}{\partial x} u(t, x). \tag{3}$$

However, lateral conditions for semigroup solutions of (3) have been given by Feller and Hille. Theorem 3 thus yields, in particular, a corresponding result on the global boundedness of the solutions of (2) (the generalized Phragmén-Lindelöf principle of P. Lax) somewhat more general than obtained hitherto in that conditions on $a(x)$ and $b(x)$ are milder, being enough to insure semigroup solutions of (3).

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