# THE CONJUGATE FOURIER-STIELTJES INTEGRAL IN THE PLANE[1]

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Let $K(x)$ with $x=(x_1, x_2)$ be a Lip $(\alpha, 2)$ conjugate Calderon-Zygmund kernel with $1/2<\alpha<1$, i.e. $K(x)=\Omega(\theta)r^{-2}$ where $(r, \theta)$ are the usual polar coordinates of $x$ with $\Omega(\theta)$ a continuous periodic function of period $2\pi$ with vanishing integral over the interval $[0, 2\pi]$ satisfying the condition $\int_0^{2\pi}[\Omega(\theta+h)-\Omega(\theta)]^2d\theta=0(h^{2\alpha})$ as $h\to 0$ (See [2] and [7, p. 106].) Let $F$ be a countably additive set function defined on the Borel sets of the plane having finite total variation. Furthermore let $f(y)=(2\pi)^{-2}\int_{E_2}e^{-i(y,x)}dF(x)$ be the Fourier-Stieltjes transform of $F$ with $E_2$ the plane and $(y, x)$ the usual scalar product. Also let $k(y)$ be the principal-valued Fourier transform of $K$, i.e. $k(y)=(2\pi)^{-2}\lim_{t\to 0;\lambda\to\infty}\int_{D(0,\lambda)-D(0,t)}e^{-i(y,x)}K(x)dx$ where $D(x, t)$ represents the open disc with center $x$ and radius $t$. (It follows from the above assumptions that $k(y)$ exists for every $y$.) Then formally the conjugate Fourier-Stieltjes integral of $F$ is given by $4\pi^2\int_{E_2}e^{i(y,x)}f(y)k(y)dy$. In [2, p. 118], it is shown that $\lim_{t\to 0}\int_{E_2-D_2(x,t)}K(x-y)dF(y)$ exists and is finite almost everywhere. We call this limit the conjugate of $F$ with respect to $K$ and designate it by $\tilde{F}(x)$. With $|y|=(y_1^2+y_2^2)^{1/2}$ and $I_R(x)=4\pi^2\int_{E_2}e^{-|y|/R}e^{i(y,x)}f(y)k(y)dy$, we propose to prove in this note the following theorem:

THEOREM 1. $\lim_{R\to\infty} I_R(x)=\tilde{F}(x)$ *almost everywhere.*

In a certain sense this result is the planar analogue of [7, p. 54]. In a forthcoming paper we shall extend this result to $n$-dimensional Euclidean space and the $n$-dimensional torus. We shall also study those kernels which are Bochner-Riesz summable almost everywhere. In particular we shall show that if $K(x)$ is in $C^\infty$ then the conjugate Fourier-Stieltjes integral of $F$ is summable $(R, \alpha)$ for $\alpha>1/2$ to $\tilde{F}(x)$ almost everywhere, thus paralleling Bochner's result [1] for the Fourier-Stieltjes integral of $F$.

Letting $D_{sym}F$ designate the symmetric derivative of $F$ [5, p.149] and $\int_B|dF|$ the total variation of $F$ over $B$, we observe from [5, p. 119 and p. 152] and the standard argument of Lebesgue that

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$$\lim_{t\to 0} (\pi t^2)^{-1} \int_{D(x,t)} |\, dF(y) - D_{sym}F(x)dy \,| = 0 \text{ a.e.} \tag{1}$$

So to prove the above theorem it is sufficient to prove the following theorem:

THEOREM 2. *At every point $x$ for which* (1) *holds*

$$\lim_{R\to\infty} \left[ I_R(x) - \int_{E_2 - D(x,R^{-1})} K(x-y)dF(y) \right] = 0. \tag{2}$$

To prove Theorem 2, we set $H_n(R) = n^{-1}\int_0^\infty e^{-t/R} J_n(t) t dt$ where $J_n(t)$ is a Bessel function of the first kind of order $n$ and establish the following lemmas:

LEMMA 1. *For $n = 1, 2, \cdots$, and all $R > 0$,*
(i) $|H_n(R)| \leqq R^2$,
(ii) $0 < H_n(R) \leqq 1$,
(iii) *there is a constant $A$ independent of $n$ and $R$ such that*

$$|H_n(R) - 1| \leqq A[(nR^{-1})^{3/2} + (nR^{-1})^{1/2}].$$

(i) follows immediately from the fact that $|J_n(t)| \leqq 1$. For $n \geqq 2$, (ii) follows on using Euler's integral representation for hypergeometric functions [6, p. 384] and [4, p. 59], for then

$$H_n(R) = \Gamma\left(\frac{n}{2}\right)\pi^{-1/2}\left[\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)\right]^{-1} \cdot \int_0^1 t^{n/2+1/2}(1-t)^{n/2-3/2}[t + R^{-2}]^{-(n/2+1)}dt.$$

(iii) follows from the fact that for $n \geqq 2$ and $R > 2$, there is a constant $A_1$ independent of $n$ and $R$ such that

$$|H_n(R) - 1| \leqq A_1 n^{1/2}\left[\int_0^{R^{-1}} t^{-1/2}dt + nR^{-2}\int_{R^{-1}}^{1/2} t^{-3/2}dt + nR^{-2}\right].$$

For $n = 1$, (ii) and (iii) follow from the fact that $H_1(R) = (1 + R^{-2})^{-3/2}$.

LEMMA 2. *Let $\Omega(\theta) = \sum_{n=1}^\infty a_n e^{in\theta} + \bar{a}_n e^{-in\theta}$ and $\alpha > \beta > 1/2$. Then there is a constant $A$ independent of $R$ and such that for $R \geqq 1$,*

$$\sum_{n=1}^\infty |\, a_n e^{in\theta} + \bar{a}_n e^{-in\theta} \,| \, |\, H_n(R) - 1 \,| < AR^{1/2-\beta}. \tag{3}$$

To prove the lemma, we observe that by [7, p. 143], $\sum_{n=1}^\infty n^{\beta-1/2} \cdot |a_n| < A_1 < \infty$ (consequently $\Omega(\theta)$ is in Lip $\beta - 1/2$), and therefore by Lemma 1 that the sum in the left part of (3) is majorized by a constant multiple of

$$R^{1/2-\beta}\left(\sum_{n=1}^{[R]} 2\,|\,a_n\,|\,n^{\beta-1/2} + \sum_{n=[R]+1}^{\infty} 4\,|\,a_n\,|\,n^{\beta-1/2}\right) \leqq 4A_1R^{1/2-\beta}.$$

To prove Theorem 2, we can assume with no loss of generality that $x$ is the origin. Next we see [3, Lemma 2] that for $y\neq 0$, $k(y) = \sum_{n=1}^{\infty} (2\pi)^{-1}(a_ne^{in\theta}+\bar{a}_ne^{-in\theta})(-i)^n n^{-1}$ and consequently that

$$I_R(0) = \int_{E_2}\left[\sum_{n=1}^{\infty}(a_ne^{in\theta} + \bar{a}_ne^{-in\theta})(-1)^nH_n(R\,|\,u\,|\,)\right]|\,u\,|^{-2}dF(u).$$

Therefore using (1), (i) of Lemma 1, and the absolute convergence of the Fourier series of $\Omega$, we obtain that

$$\tag{4} \int_{D(0,R^{-1})}\left[\sum_{n=1}^{\infty}(a_ne^{in\theta} + \bar{a}_ne^{-in\theta})(-1)^nH_n(R\,|\,u\,|\,)\right]|\,u\,|^{-2}dF(u) = o(1) \text{ as } R\rightarrow\infty.$$

Using (ii) of Lemma 1, the absolute convergence of the Fourier series of $\Omega$, and the fact that $F$ is of finite total variation on the plane, we conclude from (4) that to prove the theorem it is sufficient to show that for fixed $\lambda>0$,

$$\tag{5} \int_{D(0,\lambda)-D(0,R^{-1})}\left[\sum_{n=1}^{\infty}(-1)^n(a_ne^{in\theta} + \bar{a}_ne^{-in\theta})\,(H_n(R\,|\,u\,|\,) - 1)\right] \cdot |u|^{-2}[dF(u) - D_{sym}F(0)du] = o(1) \text{ as } R\rightarrow\infty.$$

Letting $G(t) = \int_{D(0,t)}|\,dF(u) - D_{sym}F(0)du\,|$, we see from Lemma 2 that the left side of (5) is majorized by a constant multiple of

$$\tag{6} R^{1/2-\beta}\int_{R^{-1}}^{\lambda} t^{1/2-(2+\beta)}dG(t).$$

Since by assumption $G(t) = o(t^2)$ as $t\rightarrow 0$, we obtain that (6) is $O(R^{1/2-\beta}) + o(1) + R^{1/2-\beta}\int_{R^{-1}}^{\lambda} o\,(t^2)t^{1/2-(3+\beta)}dt$. Consequently (5) is established and the proof of the theorem is complete.

## References

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