EXTENSIONS OF THE LEMMA OF HAAR IN THE CALCULUS OF VARIATIONS

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Communicated by Paul R. Halmos, June 17, 1959

This note is concerned with necessary and sufficient conditions on the coefficients A_i in order that a linear functional of the form

$$L(v) = \int_G \sum_i A_i D^i v dx \tag{1}$$

shall vanish identically on a suitable class of functions v which vanish on the boundary G^* of the connected open set G in n-dimensional x-space. Here i denotes an n-dimensional vector with nonnegative integer components i_j, and

$$D^i v = \prod_{j=1}^{n} D_{x_j}^{i_j} v,$$

where D_{x_j} denotes partial differentiation with respect to x_j. The sum in (1) is taken over all vectors i with $0 \leqq i_j \leqq m_j$, where m is a fixed vector with positive integer components.

For the domain of the functional L it is convenient to take the class of all functions v of class C^∞ and having support compact on G (i.e., compact and contained in G). Then $L(v)$ is well defined when the coefficients A_i are all locally integrable in G. Also the following notations are meaningful (with exceptional sets of measure zero) for a locally integrable function f:

$$M_{x_j h_j} f(x) = \int_0^{h_j} f(y) ds, \qquad \Delta_{x_j h_j} f(x) = f(z) - f(x),$$

where $y_j = x_j + s$, $z_j = x_j + h_j$, $y_k = z_k = x_k$ for $k \neq j$, and

$$M_h^i = \prod_{j=1}^{n} M_{x_j h_j}^{i_j}, \qquad \Delta_h^i = \prod_{j=1}^{n} \Delta_{x_j h_j}^{i_j}.$$

We understand that x is a point in G, and that h is taken so small that all the points $x + ih = (x_j + i_j h_j)$ considered lie in G. We also set

[1] This work was done under Contract No. DA-11-022-ORD-1833 with the Office of Ordnance Research. The proofs of the results reported here are contained in Technical Report No. 6 issued under this contract.

$$H_h A = \sum_i (-1)^{|i|} \Delta_h^i M_h^{m-i} A_i,$$

where $|i| = \sum_{j=1}^n i_j$.

Then the extension of Haar's lemma is as follows.

THEOREM. *The form $L(v)$ with coefficients A_i locally integrable in G vanishes for all function v in the class C^∞ and having support compact in G if and only if $H_h A = 0$ for all intervals $[x, x+h]$ contained in G except those for which one of the points $x+ih$ with $0 \leqq i_j \leqq m_j$ lies in a set E of measure 0.*

The proof of the necessity of the condition begins by observing that when the coefficients A_i are sufficiently smooth, a suitable integration by parts shows that the Euler expression

$$EA = \sum_i (-1)^{|i|} D^i A_i$$

vanishes on G. Then by use of integral means $M_h^m A / \prod_j h_j^{m_j}$ and the formula $EM_h^m A = H_h A$, we proceed to the case when the A_i are merely continuous. Finally, by another application of integral means we arrive at the general case.

Another condition for the vanishing of $L(v)$ when derivatives of order higher than the first appear was given by Hilbert in 1904 (Math. Ann. vol. 59, pp. 166–168) for the case when $n=2$ and $m_1 = m_2 = 3$, and the coefficients A_i are continuous. Extensions to other cases were proved by Mason and others. The Hilbert-Mason form of the condition may be derived from the extended form of the Haar lemma as indicated below. Assuming that the shape of the region G is suitably restricted and that a is a point of G, we set

$$I_{x_j} f(x) = \int_{a_j}^{x_j} f(t) dt_j, \quad \text{where} \quad t_k = x_k \text{ for } k \neq j,$$

$$I^i = \prod_{j=1}^n I_{x_j}^{i_j},$$

$$RA = \sum_i (-1)^{|i|} I^{m-i} A_i.$$

It is readily seen that

$$\Delta_h^m RA = H_h A,$$

and then with the help of integral means it may be shown that the condition $H_h A = 0$ almost everywhere (as stated in the theorem above) is equivalent to the following condition:

RA is equal almost everywhere in G to a sum of n functions $c_k(x)$, where $c_k(x)$ is a polynomial of degree less than m_k in x_k, with coefficients which are locally integrable functions of the remaining variables.

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