OBSTRUCTIONS TO THE SMOOTHING OF PIECEWISE-DIFFERENTIABLE HOMEOMORPHISMS

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Since the publication in 1956 of John Milnor's fundamental paper [1] in which he constructs differentiable structures on S^7 nondiffeomorphic to the standard one, several further results concerning differentiable structures have been obtained by Milnor, R. Thom, and others. This paper unifies and extends some of these results within the framework of an obstruction theory.

Two differentiable manifolds M, N (connected, not necessarily compact) will be said to be *combinatorially equivalent* if they possess isomorphic C^2 triangulations. If M and N are diffeomorphic, then they are combinatorially equivalent [5]; we seek a partial converse. Let f denote a linear isomorphism between C^2 triangulations of the n-manifolds M and N; we attempt to redefine f in neighborhoods of the open simplices of M, beginning with dimension $n-1$ and working down, so as to make f differentiable. After one step, f is no longer a linear isomorphism between C^2 triangulations; hence one must formulate more general conditions for the induction hypothesis of this step-by-step procedure. The homeomorphism f: $M\to N$ is called a *diffeomorphism mod* L, where L is the m-skeleton of a C^2 triangulation of M, if the following conditions are satisfied:

(1) f is a C^2 diffeomorphism on each closed simplex of L.

(2) f^* is one-to-one on the tangent vectors to L.

(3) The subdivision of M is fine enough that for each simplex σ of L, there are coordinate neighborhoods of $\bar{\sigma}$ and $f(\bar{\sigma})$ in which they are flat.

(4) f is of class C^1 on $M-L$, with Df bounded and $|Df|$ bounded away from zero on any subset of $M-L$ having compact closure. (Df is the Jacobian matrix.)

(5) Let σ be a simplex of L. Choose a coordinate neighborhood of $\bar{\sigma}$ in which it is flat; let z denote coordinates in the plane of $\bar{\sigma}$. Then there is a neighborhood U of σ such that

$$[\partial f/\partial z(p) - \partial f/\partial z(q)]/\|p - q\|$$

and

$$[f(p) - f(q) - Df(p)\cdot(p - q)]/\|p - q\|^2$$

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are bounded for all $q \in \sigma$ and all $p \in U - L$. The last conditions involve choice of coordinate systems; they are independent of this choice. A linear isomorphism between C^2 triangulations is a diffeomorphism mod $L = M^{n-1}$.

If f: $M \to N$ is a diffeomorphism mod L^m, we seek to approximate f by a map g: $M \to N$ which is a diffeomorphism mod L^{m-1}. There is an obstruction to the construction of g, described as follows: Consider the group Diff S^{n-1} of orientation-preserving C^1 diffeomorphisms of S^{n-1} onto itself. Those diffeomorphisms which may be extended to diffeomorphisms of the closed unit ball form a subgroup normal in Diff S^{n-1}, and the quotient is an abelian group denoted by Γ^n. Given $\sigma^m \in L$, choose coordinate systems in which $\bar{\sigma}$ and $f(\bar{\sigma})$ are flat. Let P denote an $(n-m)$-plane orthogonal to σ at $p \in \sigma$. Consider the restriction of f to P, followed by the projection of $f(P)$ into the plane orthogonal to $f(\sigma)$. In a neighborhood of p, this map g is a diffeomorphism mod p; if S_0 denotes the sphere of radius r_0 in P about p, the radial projection of $g(S_0)$ from $g(p)$ onto the unit sphere about $g(p)$ is a diffeomorphism for r_0 small. Thus there corresponds to the map f and the m-simplex σ an element of Γ^{n-m}. When one considers the orientations of the coordinate systems involved and of σ, one finds that this correspondence is a well-defined (possibly infinite) simplicial chain λf on M, with coefficients in Γ^{n-m}, providing the coefficients are twisted if M is nonorientable. λf is in fact a cycle; its homology class is called the *obstruction class* of f.

THEOREM. *Let f: $M \to N$ be a diffeomorphism* mod *the m-skeleton L of M. If $\lambda f = 0$, f may be approximated by g: $M \to N$, a diffeomorphism* mod L^{m-1}. *If $\lambda_{m-1} g$ is homologous to c_{m-1}, g may be approximated by h: $M \to N$, a diffeomorphism* mod L^{m-1}, *such that $\lambda h = c_{m-1}$.*

COROLLARY. *Let M and N be 3-dimensional differentiable manifolds. If f: $M \to N$ is a homeomorphism, f may be approximated by a diffeomorphism of M onto N.*

This follows from work of J. H. C. Whitehead [5] and E. E. Moise [2], together with the author's theorem that $\Gamma^n = 0$ for $n \leqq 3$ [3]. It was proved independently by S. Smale and the author, and by Whitehead without the approximation requirement.

COROLLARY (MILNOR). *Any two differentiable structures on euclidean space which are combinatorially equivalent are diffeomorphic.*

COROLLARY (THOM [4]). *The differentiable structures on S^n which are combinatorially equivalent to the usual one are in one-to-one correspondence with the elements of Γ^n.*

The correspondence is defined as follows: Given N combinatorially equivalent to S^n, there is a map $f: N \rightarrow S^n$ which is a diffeomorphism mod a single point p. The element of Γ^n corresponding to N is $\lambda f(p)$.

References

1. J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. vol. 64 (1956) pp. 399–405.

2. E. E. Moise, *Affine structures in 3-manifolds*, V. *The triangulation theorem and Hauptvermutung*, Ann. of Math. vol. 56 (1952) pp. 96–114.

3. J. R. Munkres, *Differentiable isotopies on the 2-sphere*, Abstract 548–137, Notices Amer. Math. Soc. vol. 5 (1958) p. 582.

4. R. Thom, *Les structures différentiable des boules et des spheres*, (to appear).

5. J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. vol. 41 (1940) pp. 809–824.

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