AN ARITHMETICAL INVERSION PRINCIPLE

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Let $f(n, r)$ represent an even function of n (mod r); that is, $f(n, r) = f((n, r), r)$ for all integers n and a positive integral variable r. The following inversion relation is proved in [2]. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(1)\qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right)\mu(d) \rightleftarrows f(n, r) = \sum_{d \mid r} g\left(d, \frac{r}{d}\right),$$

where $\mu(r)$ denotes the Möbius function. This relation can be easily verified on the basis of the definition of even function (mod r) and the characteristic property of $\mu(r)$,

$$(2)\qquad \sum_{d \mid r} \mu(d) = \epsilon(r) \equiv \begin{cases} 1 & (r = 1), \\ 0 & (r > 1). \end{cases}$$

We now state a generalization of (1). Let $\xi(r)$ and $\eta(r)$ be arithmetical functions satisfying

$$(3)\qquad \sum_{d\delta = r} \xi(d)\eta(\delta) = \epsilon(r).$$

The following theorem can be proved in the same manner as (1), with (3) used in place of (2).

THEOREM 1. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(4)\qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right)\eta(d) \rightleftarrows f(n, r) = \sum_{d\delta = (n,r)} g\left(d, \frac{r}{d}\right)\xi(\delta).$$

Clearly (4) reduces to (1) in case $\xi(r) = 1$, $\eta(r) = \mu(r)$. The case $\xi(r) = \mu(r)$, $\eta(r) = 1$ yields the following dual of (1).

THEOREM 2. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(5)\qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \rightleftarrows f(n, r) = \sum_{d\delta = (n,r)} g\left(d, \frac{r}{d}\right)\mu(\delta).$$

An immediate consequence of Theorem 2 is

COROLLARY 2.1. *For every arithmetical function* $g(r_1, r_2)$ *of two positive integral variables* r_1, r_2, *there exists a uniquely determined even function* (mod r), $f(n, r)$, *such that* $g(r_1, r_2)$ *is expressible as a divisor sum* (5) *with respect to* $f(n, r)$.

The relation (1) is applied in [2] to give a new proof of the Anderson-Apostol generalization [1] of the Hölder formula,

$$\frac{\phi(r)\mu(m)}{\phi(m)} = \sum_{d\mid(n,r)} d\mu\left(\frac{r}{d}\right), \qquad \left(m = \frac{r}{(n, r)}\right), \tag{6}$$

where $\phi(r)$ represents the Euler ϕ-function. The following analogue of the generalized Hölder relation can be proved in a similar manner, with (5) replacing (1) in the proof.

Let $g(r)$ and $h(r)$ denote arithmetical functions, and define

$$f(n, r) = \sum_{d\delta=(n,r)} h(d)g\left(\frac{r}{d}\right)\mu^2\left(\frac{r}{d}\right)\mu(\delta), \qquad F(r) = f(0, r). \tag{7}$$

THEOREM 3. *If $g(r)$ is multiplicative and $h(r)$ is completely multiplicative, and if for all primes p, $h(p)\neq 0$, $g(p)\neq h(p)$, then*

$$\frac{F(r)g(m)\mu^2(m)}{F(m)} = f(n, r), \qquad \left(m = \frac{r}{(n, r)}\right). \tag{8}$$

Application of (8), with $h(r)=r$, $g(r)=1$, in connection with the Dedekind-Liouville formula, $\phi(r) = \sum_{d\mid r} d\mu(r/d)$, yields the following analogue of Hölder's formula (6):

COROLLARY 3.1.

$$\frac{\phi(r)\mu^2(m)}{\phi(m)} = \sum_{de=r;d\delta=(n,r)} d\mu^2(e)\mu(\delta), \qquad \left(m = \frac{r}{(n, r)}\right). \tag{9}$$

Similarly, with $h(r)=1$, $g(r)=\mu(r)/\phi(r)$ in (8) it follows, on applying Landau's identity, $r/\phi(r) = \sum_{d\mid r} \mu^2(d)/\phi(d)$, that

COROLLARY 3.2.

$$\frac{(n, r)\mu(m)}{\phi(r)} = \sum_{de=r;d\delta=(n,r)} \frac{\mu(e)\mu(\delta)}{\phi(e)}, \qquad \left(m = \frac{r}{(n, r)}\right). \tag{10}$$

Other potentially useful relations can be derived in a similar manner.

BIBLIOGRAPHY

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