THE VARIATION IN INDEX OF A QUADRATIC FUNCTION DEPENDING ON A PARAMETER

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A classical theorem states that the index for $\lambda = \mu$ of the quadratic function defined by $Q(x; \lambda) = x^T(A - \lambda I)x$ is equal to the number of characteristic roots which are less than μ. In this statement, A is a symmetric matrix of constants, x is a column vector and ^{T} denotes transpose. The function Q has the following properties: (i) it is linear in λ for each x; (ii) it has negative derivative with respect to λ for each non-null x; (iii) it is positive definite for $-\lambda$ sufficiently large. This paper is an outline of a method of relaxing all three of these restrictions to the point that the dependence of Q on λ is merely smooth.

The definitions are made in a form which applies to a space of arbitrary dimension. The theorems are restricted to spaces of finite dimension, yielding both a model and a tool for an extension to spaces of infinite dimension.

The *state* of the quadratic function for a particular parameter value is described by indices which are directly defined. The changes in state are given in terms of indices of an auxiliary quadratic function (see Theorem 3). The state of the quadratic function is described alternately in terms of a characteristic value problem and the changes in state are described alternately in terms of an auxiliary characteristic value problem.

The formulation of this problem was prompted by work of M. Morse. See particularly the exposition of a lemma of Morse in §6 of the paper [**G**].

Suppose B is a symmetric bilinear function on a vector space X, depending differentiably on a parameter λ on an interval Λ for each $(x, y) \in X \times X$. Let $Q(x; \lambda) = B(x, x; \lambda)$. The *negative index* (*index* for short) of $Q(\cdot\,; \mu)$ is the least upper bound $h(\mu)$ of the dimension of planes on which $Q(\cdot\,; \mu)$ is negative definite. The *positive index* $m(\mu)$ is the index of $-Q(\cdot\,; \mu)$. The *characteristic plane* $N(\mu)$ is defined by

$$N(\mu) = \{x \mid B(x, y; \mu) = 0 \text{ for all } y \in X\}$$

and its dimension is the *nullity* $\nu(\mu)$. The value μ is a *characteristic root* if $\nu(\mu) > 0$.

The *auxiliary function* is defined by

$$q^{\mu}(x) = Q_{\lambda}(x; \mu) \mid x \in N(\mu).$$

Its index, nullity, and positive index are denoted by $h^0(\mu)$, $\nu^0(\mu)$, $m^0(\mu)$.

If the dimension n of X is finite then

$$h(\lambda) + \nu(\lambda) + m(\lambda) = n.$$

If $\nu(\lambda)$ is finite, then

$$h^0(\lambda) + \nu^0(\lambda) + m^0(\lambda) = \nu(\lambda).$$

It will now be assumed that X has finite dimension n.

THEOREM 1. *There is a positive number e such that if $\mu - e < \lambda < \mu$ then*

$$h(\mu) + m^0(\mu) \leqq h(\lambda) \leqq h(\mu) + m^0(\mu) + \nu^0(\mu) - \nu(\lambda)$$

while if $\mu < \lambda < \mu + e$ then

$$h(\mu) + h^0(\mu) \leqq h(\lambda) \leqq h(\mu) + h^0(\mu) + \nu^0(\mu) - \nu(\lambda).$$

One notes the consequence

$$0 \leqq \nu^0(\lambda) \leqq \nu(\lambda) \leqq \nu^0(\mu) \leqq \nu(\mu)$$

which holds on a sufficiently small deleted neighborhood of μ.

The quadratic function Q is called *quasi-regular* at μ if ν^0 is constant in a neighborhood of μ and *regular* at μ if $\nu^0(\mu) = 0$. Regularity implies quasi-regularity and may be easier to establish in an application, but quasi-regularity is sufficient in certain theorems. If Q is quasi-regular at each point of an interval, then ν is constant on the interval except possibly for isolated values of λ, called *primary characteristic roots*. All characteristic roots are primary if Q is regular in the interval.

THEOREM 2. *If Q is quasi-regular on an interval containing μ and if λ is on the interval and is sufficiently near μ, then for $\lambda < \mu$*

$$h(\lambda) = h(\mu) + m^0(\mu)$$

while for $\lambda > \mu$

$$h(\lambda) = h(\mu) + h^0(\mu).$$

Thus for $\lambda_1 < \mu < \lambda_2$ with $\lambda_2 - \lambda_1$ sufficiently small

$$h(\lambda_2) - h(\lambda_1) = h^0(\mu) - m^0(\mu).$$

Let

$$\sigma^0(\mu) = m^0(\mu) - h^0(\mu)$$

denote the classical signature of q^μ, which occurs in the preceding theorem.

THEOREM 3. *If Q is quasi-regular on $[\mu_1, \mu_2]$ and μ_1, μ_2 are not primary characteristic roots then*

$$h(\mu_2) - h(\mu_1) = - \sum \sigma^0(\lambda)$$

where the sum is taken for $\mu_1 < \lambda < \mu_2$.

If μ_1 or μ_2 is a primary characteristic root, the term $h^0(\mu_1)$ or $-m^0(\mu_2)$ should be added to the right hand member.

Suppose

$$Q(x; \lambda) = x^T C(\lambda) x.$$

Then μ is characteristic if and only if

$$C(\mu)x = 0 \tag{1}$$

has nontrivial solutions, and $N(\mu)$ consists of the solutions. The index $h(\mu)$ is the number of negative characteristic roots ρ of the form defined by $Q(x; \mu) - \rho x^T x$ and thus of the characteristic value problem

$$C(\mu)x - \rho x = 0. \tag{2}$$

The change in $h(\mu)$ over an interval $[\mu_1, \mu_2]$ can thus be computed by examining (2) at μ_1 and μ_2. Alternately Theorem 3 can be used. The test whether Q is regular at μ and the determination of $h^0(\mu)$ and $m^0(\mu)$ can be transformed (this is a multiplier rule) as follows.

THEOREM 4. *The index $h^0(\mu)$, the nullity $\nu^0(\mu)$, and the positive index $m^0(\mu)$ are respectively the number of negative, zero, and positive roots ρ which with $x \neq 0$ satisfy the system*

$$\begin{aligned} C(\mu)x &= 0, \\ C(\mu)y + C'(\mu)x - \rho x &= 0. \end{aligned} \tag{3}$$

That is, the determination of the difference in the state of Q at μ_1 and μ_2, or equivalently of the system (2) at μ_1 and μ_2, is replaced by the study of the system (3), which provides an alternate approach to test the applicability of Theorem 3 and to perform the computations required in it.

REFERENCE

G. M. J. Gottlieb, *Oscillation theorems for self-adjoint boundary value problems*, Duke Math. J. vol. 15 (1948) pp. 1073–1091.

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