

A REMARK ON CURVATURE AND THE DIRICHLET PROBLEM

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1. **Introduction.** Let M be a compact oriented Riemannian manifold with positive definite Ricci curvature. By a well-known theorem of Bochner-Myers [2, §26, p. 132] there are no nonzero harmonic forms of degree one on M . This result implies that "Dirichlet's problem" in the sense of §6 and §4 of reference [1] is solvable in M and that there exists the Green's form of degree one on M which is an elementary kernel for the Laplacian Δ (on 1-forms) [2, §31]. We shall point out in this note how, in this form, the result can be generalized to noncompact manifolds. We use the notations of [2].

2. **THEOREM.** *Let M be an oriented C^∞ Riemannian manifold countable at infinity. We assume that the mean curvature is positive and bounded away from zero, that is we assume that there exists a constant $C > 0$ such that*

$$R(v, v) \geq Cg(v, v)$$

for every tangent vector v , $R(v, v)$ denoting the Ricci form and $g(v, v)$, the metric form. Then Dirichlet's problem for 1-forms is solvable on M and there exists the Green's form of degree one.

PROOF. Referring to Proposition IV and §5 of [1], we have only to prove the Poincaré inequality for C^∞ 1-forms with compact supports. Let $\alpha = (\alpha_1, \dots, \alpha_k, \dots, \alpha_n)$ be a C^∞ 1-form with compact support. Then

$$(\Delta\alpha)_k = -\nabla^i \nabla_i \alpha_k - R_k^h \alpha_h.$$

We have (see [2, p. 132]),

$$\begin{aligned} (\alpha, \Delta\alpha)_{L^2} &= -\int \alpha^k \nabla_i \nabla^i \alpha_k * 1 - \int R_k^h \alpha_h \alpha^k * 1 \\ &= \int \nabla_i \alpha^k \nabla^i \alpha_k * 1 - \int R_{hk} \alpha^h \alpha^k * 1, \end{aligned}$$

using integration by parts. By assumption we have

$$-R_{hk} \alpha^h \alpha^k \geq Cg_{hk} \alpha^h \alpha^k$$

and we have

$$\int \nabla_i \alpha^k \nabla^i \alpha_k * 1 \geq 0.$$

Consequently

$$(\alpha, \Delta\alpha)_{L^2} \geq C \int g_{hk} \alpha^h \alpha^k * 1$$

that is,

$$(d\alpha, d\alpha)_{L^2} + (\partial\alpha, \partial\alpha)_{L^2} \geq C(\alpha, \alpha)_{L^2}$$

which is Poincaré's inequality for 1-forms with compact supports

REFERENCES

1. M. S. Narasimhan, *The problem of limits on a Riemannian manifold*, J. Indian Math. Soc. vol. 20 (1956) pp. 291-297.
2. G. de Rham, *Variétés différentiables*, Paris, Hermann and Cie, 1955.

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