ON SOME FUNCTIONS OF LITTLEWOOD-PALEY AND ZYGMUND

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In a previous paper [2], we studied the n-dimensional form of the functions of Littlewood-Paley and Lusin. These were defined as follows. Let $f(x)\in L^p(E_n)$, E_n is Euclidean n-space, of variables $x, y, \cdots, x=(x_1, x_2, \cdots, x_n)$; let $U(x, t)$, $t>0$, be the Poisson integral of f. Let

$$g(x) = \left(\int_0^\infty t\,|\nabla U|^2 dt\right)^{1/2} \text{ and } S(x) = \left(\iint_{W(x)} t^{1-n}\,|\nabla U|^2 dt dy\right)^{1/2}.$$

Here

$$|\nabla U|^2 = \sum_{k=0}^{n}\left(\frac{\partial U}{\partial x_k}\right)^2, \qquad x_0 = t;$$

$W(x)$ is the cone $\{(y, t):|x-y| < \alpha t\}$. We proved

$$B_p\|f\|_p \leqq \|g\| \leqq A_p\|f\|_p, \qquad 1 < p < \infty, \tag{1}$$

with a similar result for S.

We wish now to consider a related function of Littlewood-Paley and Zygmund. We define its n-dimensional version as follows. Let $0<\lambda$, and set

$$g_\lambda^*(x;f) = g_\lambda^*(x) = \left(\int_0^\infty\int_{E_n} \frac{t^{\lambda+1}}{(|x-y|^2+t^2)^{(\lambda+n)/2}}\,|\nabla U|^2 dy dt\right)^{1/2}.$$

We note first

$$g(x) \leqq AS(x) \leqq B_\lambda g_\lambda^*(x).$$

The first part of this inequality is Lemma 9 of [2], and the second part is trivial. We note also that $g^*_{\lambda_1}(x)\leqq g^*_{\lambda_2}(x)$, if $\lambda_2\leqq\lambda_1$. We shall see that the behavior of g^*_λ when $\lambda>n$ is similar to that of the simpler functions g and S. Hence our primary concern will be with g^*_λ when $0<\lambda\leqq n$. We outline the proof of the following theorem.

THEOREM. *Let $0<\lambda\leqq n$, and $2n/(\lambda+n)<p<\infty$. Then*

$$\|g_\lambda^*\|_p \leqq A_{p,\lambda}\|f\|_p, \qquad A_{p,\lambda} \textit{ independent of } f.$$

REMARKS. (i) For the one-dimensional periodic case see Zygmund [5]; for the nonperiodic case see Waterman [4]. The proofs given there are based on complex methods, which of course are unavailable in higher dimensions.

(ii) The result stated here is essentially the best possible: there exists an $f \in L^1$ so that $g_n^*(x) = \infty$, almost everywhere; also if $0 < \lambda < n$, and $p < 2n/(\lambda + n)$, there exists an $f \in L^p$, so that $g_\lambda^*(x) = \infty$, a.e.

The proof follows a series of steps.

LEMMA 1. *If* $0 < \lambda$, $2 \leqq p < \infty$, *then*

$$\|g_\lambda^*\| \leqq A_{p,\lambda} \|f\|_p.$$

In fact

$$\|g_\lambda^*\|_p^2 = \sup \int \frac{t^\lambda}{(|x-y|^2 + t^2)^{(n+\lambda)/2}} |\nabla U|^2 \phi(x) dx dy dt,$$

the sup is taken over all $\phi \geqq 0$, $\|\phi\|_r \leqq 1$, where r is the index conjugate to $p/2$. However,

$$\sup_{t>0} \int \frac{t^\lambda}{(|x-y|^2 + t^2)^{(n+\lambda)/2}} \phi(x) dx$$

$$\leqq A \sup_{t>0} t^{-n} \int_{|x| \leq t} \phi(y - x) dx = AM(\phi)(y).$$

Therefore by Fubini's theorem,

$$\|g_\lambda^*\|_p^2 \leqq A \int g^2(y) M(\phi)(y) dy \leqq A \|g\|_p^2 \|M\phi\|_r \leqq B \|f\|_p^2.$$

Here we have used inequality (1), and a well-known inequality concerning the "maximal function" $M(\phi)$.

LEMMA 2. *Let* $n < \lambda$, *then the operation* $f \rightarrow g_\lambda^*$ *is of weak type* (1, 1).

The proof of this lemma is based on the same ideas as the analogous Lemma 12 of [2], for the functions g and S.

LEMMA 3. *Let* $n < \lambda$, *then*

$$\|g_\lambda^*\|_p \leqq A_{p,\lambda} \|f\|_p, \qquad 1 < p < \infty.$$

This follows from a combination of Lemma 1 (when $n < \lambda$), Lemma 2 and the Marcinkiewicz interpolation theorem.

We can now prove the theorem. Let $\phi(x, y, t) = \phi = (\phi_0, \phi_1, \cdots, \phi_n)$ be a vector-valued function so that $\int_0^\infty \int_{E_n} |\phi(x, y, t)|^2 dy dt \leqq 1$, all x, but let ϕ be arbitrary otherwise. Let

$$T_\lambda(f)(x) = \int_0^\infty \int_{E_n} \frac{t^{(\lambda+1)/2}}{(|x-y|^2+t^2)^{(n+\lambda)/4}} \left(\sum_{k=0}^n \frac{\partial U}{\partial x_k} \cdot \phi_k \right) dy dt.$$

Then T_λ is a family of linear operators depending analytically on λ, and satisfying

$$\|T_\lambda(f)\|_p \leqq A_{p,\lambda_0}\|f\|_p, \qquad 2 \leqq p < \infty,\ R(\lambda) = \lambda_0 > 0, \tag{2}$$

$$\|T_\lambda(f)\|_p \leqq A_{p,\lambda_1}\|f\|_p, \qquad 1 < p < \infty,\ R(\lambda) = \lambda_1 > n. \tag{3}$$

The bounds A_{p,λ_0} and A_{p,λ_1} are independent of ϕ. We may now apply the convexity theorem of [1] and interpolate between (2) and (3). The result is $\|T_\lambda(f)\|_p \leqq B_{p,\lambda}\|f\|_p$, if $2n/(\lambda+n) < p < \infty$. $B_{p,\lambda}$ is independent of ϕ. Taking the sup over ϕ proves the theorem.

We shall now remark briefly on the applications of the functions g, S, and g^*_λ. The function g is basic in the Littlewood-Paley theory of Fourier series (see e.g. [7, Chapter 15]). The n-dimensional extension of these results is as yet unknown. The function S is decisive in the behavior of harmonic functions near the boundary; the n-dimensional results have recently been obtained; see [3]. An application of the function g^*_λ is one variable is given in [6]. In the following paper we shall apply the n-dimensional results to the characterization of certain classes of functions arising by "fractional integration."

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