ON A HOMOMORPHISM BETWEEN GENERALIZED GROUP ALGEBRAS

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If $G=\{a, b, \cdots\}$ is a locally compact abelian group and $X = \{x, y, \cdots\}$ a complex commutative Banach algebra, we denote by $B(G, X)$ the generalized group algebra in the sense of [**1**; **2**]. An X-valued function g defined over G is in $B(G, X)$ if g is strongly measurable and Bochner integrable with respect to Haar measure over G. We define $\|g\|_{B(G,X)} = \int_G |g(a)|_X da$ and, with convolution as multiplication, $B(G, X)$ is a complex commutative B-algebra. In [**1**, p. 1606], it is shown that the space $\mathfrak{M}(B)$ of regular maximal ideals in $B(G, X)$ is homeomorphic with $\hat{G}\times\mathfrak{M}(X)$. Here, $\hat{G}=\{\hat{a}, \hat{b}, \cdots\}$ is the character group of G and $\mathfrak{M}(X)$ denotes the space of regular maximal ideals in X, both in their usual topologies. If ϕ_M is the canonical homomorphism of X onto the complex numbers associated with an $M\in\mathfrak{M}(X)$, then a function $g\in B(G, X)$ is represented on $\mathfrak{M}(B)$ by the function $\hat{g}(\hat{a}, M) = \int_G \phi_M g(a)(a, \hat{a})da$, [**1**, p. 1604]. If $f\in L(G)$, $x\in X$, then fx shall denote the function $(fx)(a) = f(a)x$ almost everywhere over G. Clearly $fx\in B(G, X)$. Further, finite linear combinations of functions of the type fx with $f\in L(G)$, $x\in X$ are dense in $B(G, X)$.

In this paper we propose to characterize the homomorphisms T from $B(G, X)$ into $B(G, X')$ which are such that T keeps $L(G)$ "pointwise invariant." More precise statements will be found in the theorems below.

We begin with

THEOREM 1. *Let G be a group such that $\hat{G}$ is connected and let X and X' be commutative B-algebras with identities e, e' respectively. Suppose $\mathfrak{M}(X)$ is totally disconnected and X' is semi-simple. Let $T: B(G, X)\to B(G, X')$ be a continuous homomorphism such that $T(fe) = fe'$ for any $f\in L(G)$. Then there exists a continuous homomorphism $\sigma: X\to X'$ such that $(Tg)(a) = \sigma g(a)$ for any $g\in B(G, X)$.*

PROOF.[1] If $g'\in B(G, X')$ and g' is represented on its space of maximal ideals $\hat{G}\times\mathfrak{M}(X')$ as $\hat{f}\cdot\phi'$ where $f\in L(G)$ and ϕ' is a function defined on $\mathfrak{M}(X')$, then $g' = fx'$ for some $x'\in X'$. (Here, $\hat{f}(\hat{a}) = \int_G f(\hat{a})(a, \hat{a})da$.) For, consider the function F from G to X' given

[1] The author wishes to gratefully thank the referee of [1] for suggesting the following proof.

by $F(a)=\int_G g'(a)(a,\hat{a})da$. If $M'\in\mathfrak{M}(X')$, then $\phi_{M'}(F(\hat{a}))=\hat{f}(\hat{a})\phi'(M')$. If $f\neq 0$, there exists an $\hat{a}$ such that $\hat{f}(\hat{a})\neq 0$. Let $x'=F(\hat{a})/\hat{f}(\hat{a})$ so that $\phi_{M'}(x')=\phi'(M')$. We see that g' and fx' are represented by the same function on $\hat{G}\times\mathfrak{M}(X')$. Now, since X' is semi-simple, $B(G, X')$ is semi-simple [1, p. 1609] and thus $g'=fx'$.

For $\hat{a}\in\hat{G}$, $M'\in\mathfrak{M}(X')$, we have in

$$[Tg]^{\frown}(\hat{a}, M') = \int_G \phi_{M'}(Tg)(a)(a, \hat{a})da$$

a continuous multiplicative linear functional on $B(G, X)$. This means that $[Tg]^{\frown}(\hat{a}, M')=\hat{g}(\tau(\hat{a}), \sigma^*M')$ for some $\tau(\hat{a})\in\hat{G}$, $\sigma^*M'\in\mathfrak{M}(X)$. T thus induces a map $T^*: \hat{G}\times\mathfrak{M}(X')\rightarrow\hat{G}\times\mathfrak{M}(X)$ given by $T^*(\hat{a}, M') = (\tau(\hat{a}), \sigma^*M')$. Since $\hat{G}$ is connected and T^* is continuous, $T^*(\hat{G}\times\{M'\})$ is a connected set in $\hat{G}\times\mathfrak{M}(X)$ for each $M'\in\mathfrak{M}(X')$. Since $\mathfrak{M}(X)$ is totally disconnected, $T^*(\hat{G}\times\{M'\})\subset\hat{G}\times\{\sigma^*M'\}$. This is true because the connected components of $\hat{G}\times\mathfrak{M}(X)$ are precisely of the form $\hat{G}\times\{M\}$ with $M\in\mathfrak{M}(X)$. Since $T(fe)=fe'$, $f\in L(G)$, we conclude that $\tau(\hat{a})=\hat{a}$ and T^* is the product of the identity map on $\hat{G}$ and a map $\sigma^*: \mathfrak{M}(X')\rightarrow\mathfrak{M}(X)$.

Consider $fx\in B(G, X)$, $f\in L(G)$. It gets represented as a product function on $\hat{G}\times\mathfrak{M}(X)$. From the nature of T^* in the preceding paragraph, $T(fx)$ gets represented as a product function on $\hat{G}\times\mathfrak{M}(X')$ whose first factor is $\hat{f}(\hat{a})$. In view of the second paragraph of this proof, there exists $\sigma(x)\in X'$ such that $T(fx)=\sigma(x)f$. The map $\sigma: X\rightarrow X'$ is a continuous homomorphism as is easy to verify. As already remarked, finite linear combinations of functions fx are dense in $B(G, X)$ and since T is continuous, the theorem is proved.

In our next theorem G will be taken compact and we require the following

LEMMA. *Suppose G is a compact abelian group with Haar measure normalized to 1, and X is a complex commutative B-algebra with identity e with no restrictions on $\mathfrak{M}(X)$. Let ϕ be a continuous homomorphism from $B(G, X)$ to $L(G)$ which is such that $\phi(fe)=f$ for all $f\in L(G)$. Then there exists an $M\in\mathfrak{M}(X)$ such that $(\phi g)(a)=\phi_M g(a)$ a.e. for any $g\in B(G, X)$.*

PROOF. Since G is compact, the constant X-valued functions are in $B(G, X)$ and thus X can be considered to be a subset of $B(G, X)$. In other words, if $x\in X$, we denote the function $f(a)=x$ (for almost all $a\in G$) simply by x itself. If $x, y\in X\subset B(G, X)$, then $x * y=\int_G xyda = xy$ since $m(G)=1$. (Here, xy denotes the ordinary product of x and y in the B-algebra X.) Now ϕ is not identically zero on X because

$\phi(1\cdot e)=1$. Further, for any $x\in X$, $\phi(x*e)=\phi(xe)=\phi(x)=\phi(x)*\phi(e)=\phi(x)*1=\int_G\phi(x)(a)da=$ a constant a.e. over G. Hence *each* $x\in X$ *is mapped by* ϕ *onto a constant function in* $L(G)$. ϕ is additive on X and furthermore: $\phi(x*y)=\phi(xy)=\phi(x)*\phi(y)=\phi(x)\phi(y)$ for any $x, y\in X$. Consequently ϕ is a continuous nonzero multiplicative linear functional on the B-algebra X and, as such, there exists an $M\in\mathfrak{M}(X)$ with $\phi(x)=\phi_M(x)$ for $x\in X$.

Choose an arbitrary $f\in L(G)$, $x\in X$ and $\hat{a}\in\hat{G}$. We have: $\phi(fx*(\cdot,\hat{a})^{-1}e)=\phi(\hat{f}(\hat{a})(\cdot,\hat{a})^{-1}x)=\hat{f}(\hat{a})\phi((\cdot,\hat{a})^{-1}x)=\phi(fx)*\phi((\cdot,\hat{a})^{-1}e)=\phi(fx)*(\cdot,\ \hat{a})^{-1}=(\cdot,\ \hat{a})^{-1}[\phi(fx)]^{\frown}(\hat{a})$. Hence, for each $\hat{a}\in\hat{G}$, $[\phi(fx)]^{\frown}(\hat{a})=(a,\ \hat{a})\hat{f}(\hat{a})\phi((a,\ \hat{a})^{-1}x)$ for all $a\in G$ except possibly over a set of measure 0. We can therefore choose an a_0 (depending on $\hat{a}$) for which the last equation is true and substituting we find: $[\phi(fx)]^{\frown}(\hat{a})=(a_0,\ \hat{a})\hat{f}(\hat{a})\phi((a_0,\ \hat{a})^{-1}x)=(a_0,\ \hat{a})(a_0,\ \hat{a})^{-1}\hat{f}(\hat{a})\phi(x)=\hat{f}(\hat{a})\phi_M(x)$. This implies $\phi(fx)=\phi_M(x)f$ for all $f\in L(G)$, $x\in X$.

Taking finite linear combinations of functions of the type fx with $f\in L(G)$ and $x\in X$, we can find a sequence $\{f_n\}$ such that $f_n\to g$ for any $g\in B(G,\ X)$ with $\phi(f_n)=\phi_M(f_n)$. Hence $\phi(g)=\phi_M(g)$ since ϕ is continuous and the lemma is established.

THEOREM 2. *Let G and X be as in the lemma and let X' denote a semi-simple B-algebra with identity e'. Suppose $T\colon B(G,\ X)\to B(G,\ X')$ is a continuous homomorphism such that $T(fe)=fe'$ for any $f\in L(G)$. Then there exists a continuous homomorphism $\sigma\colon X\to X'$ such that $(Tg)(a)=\sigma g(a)$ for any $g\in B(G,\ X)$.*

PROOF. Let $\{W\}$ be the set of neighborhoods of the identity $0\in G$ and let $\{j_W\}$ be an approximate identity in $L(G)$. Then, if $f\in L(G)$, $x\in X$, we have $T(j_W*fx)=T(j_Wx)*fe'\to T(fx)$ as $W\to 0$. Taking Fourier transforms we find $[T(j_Wx)]^{\frown}(fe')^{\frown}\to[T(fx)]^{\frown}$ so that $[T(j_Wx)]^{\frown}(\hat{a})$ converges as $W\to 0$ for each $\hat{a}\in\hat{G}$. Call this limit $\sigma_{\hat{a}}(x)$. It is clear that $\sigma_{\hat{a}}(x)$ is independent of the approximate identity $\{j_W\}$ and the function f defining it. We have $[T(fx)]^{\frown}(\hat{a})=\sigma_{\hat{a}}(x)\hat{f}(\hat{a})e'$.

Let $M'\in\mathfrak{M}(X')$ and consider the map $\phi_{M'}\circ T$ from $B(G,\ X)$ to $L(G)$. If $fe\in B(G,\ X)$ with $f\in L(G)$, then $(\phi_{M'}\circ T)(fe)=\phi_{M'}(fe')=f$. The map $\phi_{M'}\circ T$ is continuous and the lemma applies to it. Therefore, there is an $M\in\mathfrak{M}(X)$, depending on $M'\in\mathfrak{M}(X')$, such that $\phi_{M'}\circ T=\phi_M$. Now: $\phi_{M'}[T(fx)]^{\frown}(\hat{a})=[(\phi_{M'}\circ T)(fx)]^{\frown}(\hat{a})=\hat{f}(\hat{a})\phi_M(x)=\hat{f}(\hat{a})\phi_{M'}(\sigma_{\hat{a}}(x))$. This means $\phi_{M'}(\sigma_{\hat{a}}(x))=\phi_M(x)$ for each $M'\in\mathfrak{M}(X')$ and each $\hat{a}\in\hat{G}$, $x\in X$.

We show that $\sigma_{\hat{a}}(x)$ is actually independent of $\hat{a}$. Suppose $\sigma_{\hat{a}_1}(x)=y_1$, $\sigma_{\hat{a}_2}(x)=y_2$ and $\hat{a}_1\neq\hat{a}_2$. For any $M'\in\mathfrak{M}(X')$ we have $\phi_{M'}(y_1)=\phi_{M'}(y_2)=\phi_M(x)$. Since X' is semi-simple, we must have $y_1=y_2$ and so $\sigma_{\hat{a}}(x)$ is independent of $\hat{a}$. Write $\sigma_{\hat{a}}(x) = \sigma(x)$. σ is a continuous homomorphism from X to X'. We have, consequently, shown that $[T(fx)]^\frown(\hat{a}) = \sigma(x)\hat{f}(\hat{a})$ and this means $T(fx) = \sigma(x)f$ for all $f \in L(G)$, $x \in X$, because $B(G, X')$ is semi-simple if X' is semi-simple. Continuing in a manner like that at the end of the lemma or the end of Theorem 1, we find that $(Tg)(a) = \sigma g(a)$ for all $g \in B(G, X)$. This completes the proof.

We remark that, conversely, if $\sigma: X \to X'$ is a continuous homomorphism, then the map $(Tg)(a) = \sigma g(a)$ from $B(G, X)$ to $B(G, X')$ is a continuous homomorphism with no restrictions on G, $\hat{G}$, X or X'. The proof is easy and is omitted.

THEOREM 3. *In either Theorem* 1 *or* 2, *if* T *is an isomorphism from* $B(G, X)$ *onto* $B(G, X')$, *then* σ *is an isomorphism from* X *onto* X'.

PROOF. σ is one-one for if $x_1 \neq x_2$, $x_1, x_2 \in X$, then $fx_1 \neq fx_2$ where $f \in L(G)$, $f \neq 0$. Since T is one-one, $T(fx_1) = \sigma(x_1)f \neq T(fx_2) = \sigma(x_2)f$ so that $\sigma(x_1) \neq \sigma(x_2)$. σ is onto X', for choose any $x' \in X'$. Find an $f \in L(G)$ such that $\hat{f}(\hat{0}) \neq 0$. Since T is onto, there is a $g \in B(G, X)$ such that $\sigma g = fx'$. Taking Fourier transforms: $[\sigma g]^\frown(\hat{a}) = \sigma\hat{g}(\hat{a}) = \hat{f}(\hat{a})x'$. Setting $\hat{a} = \hat{0}$, we find $\sigma\hat{g}(\hat{0}) = \hat{f}(\hat{0})x'$ so that $\sigma(\hat{g}(\hat{0})/\hat{f}(\hat{0})) = x'$. Hence, there is an $x = \hat{g}(\hat{0})/\hat{f}(\hat{0}) \in X$ such that $\sigma(x) = x'$.

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