ON THE IRREDUCIBILITY OF CERTAIN MULTIPLIER REPRESENTATIONS

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Communicated by A. M. Gleason, October 24, 1961

If M is a complex manifold and G is a transitive group of holomorphic automorphisms of M, there is a natural unitary representation, T on G of the Hilbert space of square integrable holomorphic forms of maximal rank on M. Kobayashi has given a very simple and elegant proof that T is irreducible, [2]. It is the purpose of this note to show that the main idea of Kobayashi's proof can be applied in the related but somewhat different context of multiplier representations.

THEOREM. *Let M be a (connected) complex manifold and G a transitive group of holomorphic automorphisms of M. Suppose that H is a Hilbert space of holomorphic functions on M in which point evaluations are nonzero continuous linear functionals. Then every unitary multiplier representation of G on H is irreducible.*

Let $T\colon a \rightarrow Ta$ be a given representation of G on H. We shall say that T is a multiplier representation if there is a function m defined on $M \times G$ such that

$$(Taf)(z) = m(z, a)f(za), \qquad f \in H \tag{1}$$

where za denotes the image of z under the transformation a.

The crucial analytic point in the proof is supplied by the following result.

LEMMA. *Let M be a (connected) complex manifold and h a function on $M \times M$ with the following properties:*

(a) $h(z, z) = 0$, *all* $z \in M$

(b) *for fixed z, $h(z, w)$ is holomorphic as a function of w and conjugate holomorphic as a function of z, w being fixed. Then $h(z, w) = 0$ for all z and w in M.*

This may be proved by considering local power series expansions; these exist by a variant of Hartog's theorem. Actually, in our application, the uniform boundedness principle shows that the function in question is bounded locally so that the full strength of Hartog's theorem is not necessary.

[1] Research supported in part by National Science Foundation Research Grant NSF G-12847.

PROOF OF THE THEOREM. Our assumptions on H show that for each z in M there exists a unique nonzero element k_z of H such that

$$f(z) = (f, k_z), \qquad f \in H \tag{2}$$

where (f, g) denotes the inner product of f and g, [1]. Thus $\|k_z\|^2 = k_z(z) > 0$ and

$$k_z(w) = \bar{k}_w(z) \tag{3}$$

for all points z and w of M.

We now assume that T is a representation of G on H and that it is given by (1). Since $Tab = TaTb$, i.e. since

$$m(z, ab)f(zab) = m(z, a)m(za, b)f(za \cdot b)$$

for all f in H, it follows that m satisfies

$$m(z, ab) = m(z, a)m(za, b). \tag{4}$$

Next we assume that T is unitary. Then $(Tak_z, k_w) = (k_z, Ta^{-1}k_w)$ so that $m(w, a)k_z(wa) = \bar{m}(z, a^{-1})\bar{k}_w(za^{-1})$ for all z and w in M. Replacing z by za and using (3) and (4) we find that

$$k_z(w) = \bar{m}(z, a)m(w, a)k_{za}(wa). \tag{5}$$

Now let H' be any nonzero closed subspace of H invariant under T and k_z' the orthogonal projection of k_z on H'. Then k_z' satisfies obvious analogues of (2) and (3), and hence

$$k_z'(w) = \bar{m}(z, a)m(w, a)k_{za}'(wa) \tag{6}$$

as well. Since $H' \neq 0$, it follows from (5) and (6) and the transitivity of G that

$$k_z(z) = ck_z'(z), \qquad z \in M \tag{7}$$

for some $c \geqq 1$. For fixed z, $k_z(w)$ is holomorphic as a function of w and by (3) conjugate holomorphic as a function of z, w being fixed. Since $k_z'(w)$ has similar properties it follows from (7) and the lemma that $k_z(w) = ck_z'(w)$ for all z and w. Thus $H' = H$, as any f orthogonal to H' is orthogonal to k_z for every z and hence is equal to 0, by (2). This shows that T is irreducible and completes the proof.

REFERENCES

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