A ZERO-ONE PROPERTY OF MIXING SEQUENCES OF EVENTS[1]

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A sequence of events $\{A_n\}_1^\infty$ is *mixing* if for each event $M, P(A_nM) - P(A_n)P(M) \to 0$; if it is also assumed that $P(A_n) \to \alpha$, $0 \leq \alpha \leq 1$, $\{A_n\}$ is called *mixing with density* α. A sequence of events $\{A_n\}$ is *zero-one* if its tail is trivial; *semi-zero-one* if every subsequence of $\{A_n\}$ admits a zero-one subsequence.

THEOREM 1. *A sequence of events is mixing if and only if it is semi-zero-one.*

OUTLINE OF PROOF. Denote by $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$, respectively, the σ-fields generated by the event A_n, by the events $A_1, \cdots, A_n$, and by the events $A_n, A_{n+1}, \cdots$; $\mathcal{C} = \bigcap_1^\infty \mathcal{C}_n$ is the *tail* of the sequence $\{A_n\}$. If $\{A_n\}$ is zero-one, then for every bounded random variable X $E(X/\mathcal{C}_n) \to E(X)$ with probability one and in L_1 mean. One shows that if $P(A_n) \to \alpha$, $0 < \alpha < 1$, then $\{A_n\}$ is mixing if and only if, for each bounded random variable X, $E(X/\mathcal{A}_n) \to E(X)$ in L_1 mean ("in L_1 mean" may here be replaced by "in probability" or by "uniformly except on a null event"). Hence a zero-one sequence, and also a semi-zero-one sequence, will be mixing. Now denote by A^v the event A or its complement and by I_A the characteristic function of A. Let $\{A_n\}$ be a sequence such that all events $A_1^v \cdots A_n^v$, $n = 1, 2, \cdots$, are not null and let Q be the independent probability measure on $\mathcal{C}_1$ with $Q(A_n) = \alpha$, $n = 1, 2, \cdots$, $0 < \alpha < 1$. Set

$$X_n = \sum I_{A_1^v \cdots A_n^v} \frac{P(A_1^v \cdots A_n^v)}{Q(A_1^v \cdots A_n^v)}, \qquad n = 1, 2, \cdots, \tag{1}$$

where the summation extends over all events $A_1^v \cdots A_n^v$ of $\mathcal{B}_n$. It is shown that every sequence of events mixing with density α admits a subsequence $\{A_n\}$ such that the X_n's defined by (1) are uniformly integrable with respect to the measure Q (even uniformly bounded by $1-\epsilon$, $1+\epsilon$ where ϵ is arbitrarily small). Doob's discussion [1, pp. 343 ff.] shows that P is absolutely continuous with respect to Q on $\mathcal{C}_1$; by Kolmogorov's zero-one law $\{A_n\}$ is Q zero-one, hence $\{A_n\}$ is also P zero-one. It follows that a mixing sequence is semi-zero-one,

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unless it is mixing with density zero or one in which case one shows the existence of a zero-one subsequence by a direct argument.

From Theorem 1, one obtains that Kolmogorov automorphisms are mixing in all degrees [cf. Rokhlin **2**, p. 14]. In probabilistic formulation: discrete-parameter stationary stochastic processes with trivial tail are mixing in all degrees. (The tail of a process $\{X_n\}_{-\infty}^{\infty}$ is the σ-field $\bigcap_{-\infty}^{\infty} \mathcal{C}_n$ where $\mathcal{C}_n$ is generated by $\cdots X_{n-1}, X_n$.) Via distribution functions Theorem 1 may also be applied to processes not necessarily stationary. A sequence of random variables $\{X_n\}_1^{\infty}$ is called *mixing* if for some dense set D on the real line R the sequence of events $\{A_n(y)\}$ is mixing for each $y \in D$, where $A_n(y)$ is defined on R by

$$A_n(y) = [X_n < y], \qquad n = 1, 2, \cdots.$$

If $\{A_n(y)\}$ is mixing with density $\bar{F}(y)$ for $y \in D$, then $\bar{F}$ determines a distribution function F defined on R and $P(A_n(y))$ converges to $F(y)$ on the continuity set of $F(y)$; the sequence of random variables $\{X_n\}$ is then called *mixing with the limiting distribution function* $F(y)$; this last notion was introduced by Rényi [**3**]. It follows from Theorem 1 that if a sequence of random variables $\{X_n\}$ is *semi-zero-one*, i.e. if every subsequence contains a subsequence with trivial tail, then $\{X_n\}$ is mixing. It is further shown under rather weak assumptions that mixing is invariant under change of measure. A probability measure Q is *semicontinuous* with respect to P on a sequence of random variables $\{X_n\}$ if every subsequence of $\{X_n\}$ contains a further subsequence $\{Y_n\}$ such that Q is absolutely continuous with respect to P on the tail of $\{Y_n\}$.

THEOREM 2. *Let a sequence of random variables $\{X_n\}$ be P mixing (with a limiting distribution function $F(y)$). If Q is a probability measure semicontinuous with respect to P on $\{X_n\}$, then the sequence $\{X_n\}$ is Q mixing (with the limiting distribution function $F(y)$).*

In the proof, the invariance of mixing is obtained from Theorem 1 while the invariance of the limiting distribution is derived from the second theorem of Andersen and Jessen [**4**].

Theorem 2 extends Theorem 2 of Abbot and Blum [**5**] and certain results on invariance of limiting distributions of Rényi and Révész. Namely in Theorem 4 of Rényi [**3**] concerned with sums of independent random variables and in Examples 3 and 4 of Rényi and Révész [**6**] concerned with certain Markov chains, the premises may be weakened by assuming semicontinuity of Q with respect to P on the studied sequences of averages of random variables, instead of absolute continuity of Q with respect to P on $\mathfrak{A}$; the conclusions may be strengthened by asserting Q mixing of these sequences with the limiting distribution function $F(y)$, instead of only the convergence of the distribution functions of the averages to $F(y)$.

BIBLIOGRAPHY

1. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.

2. V. A. Rokhlin, *New progress in the theory of transformations with invariant measure*, Uspehi Mat. Nauk (1960), English translation published by the London Math. Soc.

3. A. Rényi, *On mixing sequences of sets*, Acta Math. Acad. Sci. Hungar. **9** (1958), 215–228.

4. E. S. Andersen and B. Jessen, *Some limit theorems on set-functions*, Danske Vid. Selsk. Mat.-Fys. Medd. **25** (1948).

5. J. H. Abbott and R. J. Blum, *On a theorem of Rényi concerning mixing sequences of sets*, Ann. Math. Statist. **32** (1961), 257–260.

6. A. Rényi and P. Révész, *On mixing sequences of random variables*, Acta Math. Acad. Sci. Hungar. **9** (1958), 389–393.

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THE EQUATION $(\partial^2/\partial x^2+\partial^2/\partial y^2+(x^2+y^2)(\partial/\partial t))^2u+\partial^2u/\partial t^2=f$, WITH REAL COEFFICIENTS, IS "WITHOUT SOLUTIONS"

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Indeed, the equation can be written $PP^*(PP^*)^*u=f$, where P is Lewy's operator $\partial/\partial\bar{z}+iz(\partial/\partial t)$,[2] $z=x+iy$, and the star operation replaces the coefficients of a differential operator by their complex conjugates. Hörmander has shown[3] that, whatever be the open set Ω, there is a function $f\in C_0^\infty(\Omega)$ such that the equation $Pv=f$ does not have any distribution solution $v\in\mathfrak{D}'(\Omega)$.

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[2] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155.

[3] L. Hörmander, *Differential equations without solutions*, Math. Ann. **140** (1960), 169.