ON THE QUOTIENT OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

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The origin of this note is a question in the theory of functions: "If $f_1(z)$ and $f_2(z)$ are two entire functions of lower order less than one and if $f_1(z)$ and $f_2(z)$ have the same zeros, is $f_1(z)/f_2(z)$ a constant?" This is one of 25 problems published in Bulletin of the American Mathematical Society, January 1962, pp. 21–24.

The solution of this problem is that the quotient $f_1(z)/f_2(z)$ is not necessarily a constant. It is even possible to find such entire functions of lower order zero. To do this we introduce some definitions.

$$a_n = 2^{(4n)!}, \qquad b_n = 2^{(4n+2)!}, \qquad P_n(z) = \left(1 - \frac{z}{a_n}\right)^{a_n}\left(1 + \frac{z}{b_n}\right)^{b_n},$$

$$f_1(z) = \prod_{n=1}^{\infty} P_n(z), \qquad f_2(z) = e^{-z} f_1(z).$$

Now $f_1(z)$ and $f_2(z)$ are different entire functions with the same zeros. We denote

$$M_v(r) = \max_{|z|=r} |f_v(z)|, \qquad v = 1, 2.$$

For $r = 2^{(4m+3)!}$ the rough estimate

$$\frac{\log \log M_1(r)}{\log r} < \frac{1}{m}$$

implies that $f_1(z)$ is of lower order zero.

For $r = 2^{(4m+1)!}$ we obtain

$$\frac{\log \log M_2(r)}{\log r} < \frac{1}{m}$$

which implies that $f_2(z)$ is of lower order zero.

Thus the problem is solved.

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