ON REAL JORDAN ALGEBRAS

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Let X be a vector space of the finite dimension n over the field R of the real numbers. For a (scalar or vector valued) function f defined in a neighbourhood of $x \in X$ and differentiable in x, the operator

$$\Delta_x^u f(x) = \frac{d}{d\tau} f(x + \tau u) \Big|_{\tau=0}$$

is defined and linear for every $u \in X$.

We consider triples (Y, ω, c) fulfilling the following conditions:

(i) Y is an open and connected subset of X such that $\lambda > 0$ and $y \in Y$ implies $\lambda y \in Y$.

(ii) $\omega = \omega(y)$ is a continuous real-valued function on the closure $\overline{Y}$ of Y that is homogeneous of degree n, positive, and real analytic in Y, and vanishes on the boundary of Y. Furthermore, the Hessian $\Delta_y^u \Delta_y^v \log \omega(y)$ is nonsingular for $y \in Y$.

Let c be a given point in Y and denote by $\sigma(u, v)$ the Hessian of $\log \omega(y)$ at the point $y = c$. Without restriction we may assume that $\omega(c) = 1$ holds. Since $\sigma(u, v)$ is nonsingular, the adjoint transformation A^* (with respect to σ) is defined for every linear transformation A of X. We form the group Σ' of those linear transformations W of X for which $y \to Wy$ is a bijective map of Y onto itself and for which $\omega(Wy) = \|W\| \omega(y)$ holds identically for $y \in Y$. Here $\|W\|$ denotes the absolute value of the determinant of W. Let Σ be the subgroup of Σ' consisting of the transformations W in Σ' for which $W^* \in \Sigma'$ holds. The triple (Y, ω, c) is called an *Ω-domain*, if (i), (ii) hold and in addition

(iii) Σ acts transitively on Y.

On the other hand, we consider in X a *Jordan algebra*, i.e., a bilinear and commutative composition $(x, y) \to xy$ of $X \times X \to X$ fulfilling

$$x^2(xy) = x(x^2 y)$$

for every x, y in X. Such a Jordan algebra, that is, the vector space X together with the composition, shall be denoted by A. For every $x \in X$ the mapping $y \to xy$ determines a linear transformation $L(x)$ of X such that $xy = L(x)y$. Denote by $\tau(x, y)$ the trace of $L(xy)$. Then $\tau(x, y)$ is a symmetric bilinear form on X. The Jordan algebra A is called *semi-simple* if $\tau(x, y)$ is nonsingular. It is known, that a semi-simple Jordan algebra contains a unit element c. Besides the linear transformation $L(x)$ there is another, more important transformation $P(x)$ introduced by N. Jacobson [1] that is defined by

$$y \to P(x)y = 2x(xy) - x^2y, \quad \text{i.e.,} \quad P(x) = 2L^2(x) - L(x^2).$$

$P(x)$ fulfills the following identity

$$P(P(x)y) = P(x)P(y)P(x), \qquad x, y \in X. \tag{1}$$

A first proof of this formula (in case of semi-simple real Jordan algebras) can be found in Ch. Hertneck [5]. The proof for general Jordan algebras was given independently by I. G. MacDonald [6]. Since $P(c)$ is the identity mapping, the determinant $|P(x)|$ is not identically zero. The transformation $P(x)$ is called the *quadratic representation* of the Jordan algebra A. Formula (1) shows for instance

$$P(x^m) = P^m(x) \qquad \text{for } m = 1, 2, \cdots.$$

However, $P(x)$ is not a representation in the sense, that $x \to P(x)$ is a homomorphism of A into the Jordan algebra of linear transformations.

To a given semi-simple Jordan algebra A we define a triple (Y_A, ω_A, c_A), denoted by $\Omega(A)$, in the following way: c_A is the unit element of A, $\omega_A(x) = \|P(x)\|^{1/2}$ and Y_A is the connected component of the set $\{x; \omega_A(x) \neq 0\}$ containing the point c_A.

Using certain results on Jordan algebras, in particular formula (1), the theory of eigenvalues of a Jordan algebra (following ideas of E. Artin) and the notion of the inverse in a Jordan algebra (due to N. Jacobson [1], following a representation of E. Artin), we are able to prove

THEOREM 1. *Let A be a semi-simple Jordan algebra over R. Then $\Omega(A)$ is an Ω-domain in the vector space underlying A. The bilinear form σ associated with the Ω-domain coincides with the bilinear form τ of A. Moreover, the transformations $P(x)$, $|P(x)| \neq 0$, belong to the group Σ which is associated with the Ω-domain.*

In the course of the proof it turns out, that even the group generated by the transformations $P(x)$ where x varies in some neighbourhood of the unitelement c_A, acts transitively on Y_A.

Vice versa, let us start out with an Ω-domain (Y, ω, c) in X. An investigation of the geodesics with respect to the (in general not positive definite) metric given by the Hessian $\Delta_y^u \Delta_y^v \log \omega(y)$ leads to

THEOREM 2. *Let (Y, ω, c) be an Ω-domain in X. Then there exists a semi-simple Jordan algebra A in X such that $(Y, \omega, c) = \Omega(A)$.*

Furthermore we get

THEOREM 3. *The map $A \to \Omega(A)$ of the family of semi-simple real Jordan algebras A is a bijection onto the family of Ω-domains.*

It is important to know under which circumstances two semi-simple Jordan algebras give rise to Ω-domains that are not essentially different. We call two Ω-domains (Y, ω, c) resp. (Y', ω', c') defined over the vector space X resp. X' of the same dimension, *equivalent* if there is a bijective linear transformation $V: X \to X'$ such that

$$Y' = VY, \qquad \omega'(Vy) = \gamma \cdot \omega(y) \qquad \text{for } y \in Y$$

holds, where γ is a suitable real number. Then we get

THEOREM 4. *Two Ω-domains $\Omega(A)$ and $\Omega(B)$ are equivalent if and only if the Jordan algebras A and B are isomorphic.*

Given a Jordan algebra A and $f \in A$. Then one can define a new multiplication in the underlying vector space X by

$$x \perp y = x(yf) + y(xf) - (xy)f.$$

X together with the composition $\perp$ shall be denoted by A_f. It is known that A_f is a Jordan algebra. The quadratic representation of A_f turns out to be $P(x)P(f)$, where $P(x)$ is the quadratic representation of A.

Given a semi-simple Jordan algebra A, let us consider the subset X_A of the underlying vector space X consisting of all points x for which $|P(x)| \neq 0$. The connected component of the unitelement of A is an Ω-domain (see Theorem 1). In addition we get

THEOREM 5. *Let C be a connected component of X_A and $f \in C$, then the triple $(C, \|P(f)\|^{1/2} \cdot \omega_A, f^{-1})$ is the Ω-domain, which is associated with the semi-simple Jordan algebra A_f.*

Here f^{-1} denotes the inverse of f in the Jordan algebra A. Combining Theorems 4 and 5 we have

THEOREM 6. *There is a one-to-one correspondence between the equivalence classes of connected components of X_A (considered as Ω-domains) and the isomorphic classes of the Jordan algebras A_f where $f \in X_A$.*

Special cases of Ω-domains are the homogeneous *domains of positivity* (see [**2**; **3**; **4**], O. S. Rothaus [**7**], Ch. Hertneck [**5**] and E. B. Vinberg [**8**]). It is known, that the map $A \to \Omega(A)$ maps the family of formal real Jordan algebras onto the family of homogeneous domains of positivity. Here a Jordan algebra A is called *formal real* if $x^2 + y^2 = 0$ implies $x = y = 0$. This is equivalent to the notion of a *compact* Jordan algebra, i.e., a Jordan algebra, for which the bilinear form $\tau(x, y)$ is positive definite. This gives an algebraic characterization of the Jordan algebras associated with domains of positivity. However, there is a different geometric characterization of the domains of positivity in the family of Ω-domains.

THEOREM 7. *An Ω-domain (Y, ω, c) is an homogeneous domain of posivitity if and only if the set Y is convex.*

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