POLYNOMIALLY CONVEX SETS

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0. **Introduction.** This is a report on the *polynomial convex hull* of a compact set, X, in complex n-space, C^n. By definition, it is hull(X), *the set of all* p *in* C^n *such that*

$$|f(p)| \leqq \max_{x\in X} |f(x)|,$$

for every polynomial, $f(z_1, \cdots, z_n)$. When $X = \text{hull}(X)$, we say X *is polynomially convex*. In studying the polynomial convex hull, it helps to introduce also R-hull(X), the *rational convex hull* of X. By definition, R-hull(X) is *the set of all* p *in* C^n *such that*

$$|g(p)| \leqq \max_{x\in X} |g(x)|,$$

for all rational functions, g, *which are analytic about* X. Equivalently, R-hull(X) may be described as the set of all p in C^n for which $f(p)$ belongs to $f(X)$, for every polynomial, f. When $X = R\text{-hull}(X)$, we say that X *is rationally convex*. Evidently,

$$X \subset R\text{-hull}(X) \subset \text{hull}(X),$$

and these hulls are compact.

Polynomially convex sets occur prominently in the theory of uniform approximation. Every finitely-generated function algebra can be realized as the uniform closure of the polynomials on a compact subset, X, of some C^n; in which case, its maximal ideal space is precisely hull(X) [**13**].

I. **Local descriptions of the hulls.** To begin, we explain what we mean by a curve of analytic hypersurfaces in an open subset, O, of C^n. If U is a domain in O and F_t, $0 \leqq t < 1$, is a curve of nonconstant analytic functions on U, we let H_t be the zero-set of F_t in U. If each H_t is closed in O, then we say that (F_t, H_t) *is a curve of analytic hypersurfaces in* O. We shall denote it, simply, by (H_t).

Almost all the results of §§I and II are based on the following beautiful local characterization of the polynomial convex hull, given by K. Oka in 1937, in [6].

(I.0) OKA'S CHARACTERIZATION THEOREM. *Let* O *be a neighborhood of* hull(X) *in* C^n. *If* (H_t) *is a curve of analytic hypersurfaces in* O *such* *that some* H_t *intersects* hull(X), *but some other* $H_{t'}$ *does not, then some* $H_{t''}$ *must intersect* X.

Oka's proof uses both his solution of the Cousin I problem [5], and the Oka-Weil Approximation Theorem [5], as a way of passing, in two steps, from local to global information.

Theorem I.0 yields very easily

(I.1) ROSSI'S LOCAL MAXIMUM MODULUS PRINCIPLE. *If* Y *is a subset of* hull(X) $-X$, *then* Y *is contained in*[1] hull(∂Y) [7]. Moreover, by analyzing the Cousin I problem solved in the proof of Theorem I.0 and appealing to E. Bishop's "$1/4-3/4$" description of peak points,[2] [1], we obtain also

(I.2) ROSSI'S LOCAL PEAK POINT THEOREM. *On* hull(X), *every local peak point*[2] *is a peak point* [7].

In the original proofs of Theorems I.1 and I.2, Rossi used a somewhat more difficult argument, based on a solution of a Cousin II problem,[3] (see [7]), and obtained somewhat stronger results. It happens that, for the corresponding local description of the rational convex hull (Theorem I.3 below), a Cousin II type argument is essential.

(I.3) THEOREM. *If* Y *is a subset of* R-hull(X) $-$ X, *and if*[4] $\check{H}^2(R\text{-hull}(X); Z)=0$, *then* Y *is contained in* R-hull(∂Y).

The topological restriction is really needed. It can be shown, by an example, that the conclusion of Theorem I.3 may not obtain if $\check{H}^2(R\text{-hull}(X); Z)\neq 0$.

II. **Applications to simply-coconnected sets.** We shall say that X *is simply-coconnected* when[5] $\check{H}^1(X; Z)=0$. If X is a compact subset of C^n, we say that X *is polynomially convex in dimension one* if[6] $X\cap V$ is polynomially convex, for every V which is a complex one-dimensional analytic subvariety of C^n.

(II.0) REMARK. *Every simply-coconnected, rationally convex set is polynomially convex in dimension one.* However, there do exist simply-coconnected sets which are polynomially convex in dimension one, but which are not rationally convex. (Consider, for example, the non-rationally convex arc described by J. Wermer in [9].)

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[1] ∂ denotes *topological boundary* (in the hull of X).

[2] As defined in [7, p. 6], for the uniform closure of the polynomials on hull(X).

[3] However, A. Browder noticed that a Cousin I solution would suffice in I.1.

[4] Čech cohomology with integer coefficients.

[5] Equivalently, every map of X into $C^1-\{0\}$ has a log.

[6] This is a semi-topological condition, asserting that every component of $V-(X\cap V)$ is unbounded.

All the results of this section will depend on the following

(II.1) THEOREM. *Let X be a compact set in C^n and let f be a polynomial. If a branch of* $\log(f)$ *is defined on X, and $f(X)$ does not intersect* $f(\text{hull}(X)-X)$, *then f does not vanish at any point of* $\text{hull}(X)$.

The proof is based on Oka's Characterization (I.0). Roughly, if f were to vanish at a point of hull(X), then $\log(f)$ would "unwind" X from hull(X). In that case, we study the image, under $\log(f)$, of a neighborhood of X, and construct a certain curve of analytic hypersurfaces which would contradict Oka's Characterization.

A direct corollary is

(II.2) THEOREM. *If X is a compact, simply-coconnected subset of C^n, and if there is a polynomial, f, such that $f(X)$ does not intersect* $f(\text{hull}(X)-X)$, *then X is polynomially convex.*

We put our most general application of Theorems II.1 and II.2 in the following way.

(II.3) THEOREM. *Let X be a compact subset of C^n which is simply-coconnected and polynomially convex in dimension one. Suppose there are $n-1$ polynomials, $f_1, \cdots, f_{n-1}$, which are in general position*[7] *on C^n, and such that, for each $j=1, \cdots, n-1$, $f_j(X)$ is contained in the minimal boundary*[8] *for the uniform limits of functions analytic about* $f_j(\text{hull}(X))$. *Then X is polynomially convex.*

We next list some very special cases of Theorem II.3. But first, define *the n-torus*, $T(n)=\{p\in C^n: |z_i(p)|=1, i=1, \cdots, n\}$. Then,

(II.4) THEOREM. *If X is simply-coconnected and rationally convex (or, if X is an arc) and lies in $T(n-1)\times C^1$, then X is polynomially convex.*

Since every subset of $T(n)$ is rationally convex, this implies

(II.5) THEOREM. *If X is contained in a simply-coconnected subset of $T(n)$, then X is polynomially convex. Moreover, every complex-valued continuous function on X is a uniform limit of polynomials.*

Also, we have

(II.6) THEOREM. *If X is simply-coconnected and rationally convex (or, if X is an arc) and lies in C^2, and if there is a nonconstant polynomial, f, with $|f|=1$ on X, then X is polynomially convex.*

REMARK. We would be delighted to know the answer to the

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[7] Their common level sets have complex dimension, at most, one.

[8] Defined in [1; 13]. In many cases it is identical with $\partial(f_j(\text{hull}(X)))$.

(II.7) QUESTION. *Is every simply-coconnected, rationally convex set polynomially convex?*

Theorem II.2 is as close as we have come to an affirmative answer. (But see also Remark III.3.)

III. **Approximation by analytic polyhedra.** By *an analytic polyhedron, P, in C^n,* we shall mean *a polynomially convex set of the form*

$$P = \{u \in U: |f_j(u)| \leqq k_j, j = 1, \cdots, r\},$$

where U is an open subset of C^n, the f_j are analytic on U, and the k_j are non-negative constants.

It is evident, from the very definition of a polynomially convex set, that it can be expressed as a decreasing intersection of analytic polyhedra (where the defining functions are, in fact, polynomials). A far deeper result of E. Bishop, [**2**, p. 225], shows that, in C^n, we can arrange that every approximating polyhedron be defined by exactly n inequalities.

By the general theory of function algebras, [**13**], it is known that, for every compact subset, X, of C^n, there exists S_X, the unique smallest closed subset for which $\text{hull}(S_X) = \text{hull}(X)$. We call S_X *the Šilov boundary of* $\text{hull}(X)$. For an analytic polyhedron, P, in C^n, defined by precisely n inequalities, $|f_j| \leqq 1$, $j = 1, \cdots, n$, the Šilov boundary, S_P, has an especially pleasing form.

$$S_P = \{u \in U: |f_j(u)| = 1, j = 1, \cdots, n\}.$$

Thus, it "lies over" the n-torus.

A number of people have recognized the value of an affirmative answer to the following

(III.0) QUESTION. *If L is a polynomially convex set, is it always possible to find L_i, a sequence of analytic polyhedra, converging*[9] *to L, in such a way that the Šilov boundaries, S_{L_i}, converge to S_L?* Unfortunately, the answer is "no." This will be a consequence of the next few results.

(III.1) THEOREM. *Let L be an analytic polyhedron. If h is analytic about L and* $\log(h)$ *is defined on S_L, then h does not vanish at any point of L.*

Moreover, this persists in the limit. That is,

(III.2) COROLLARY. *Let L_i be a sequence of analytic polyhedra converging to a compact set, L. Let S be the limit set of the sequence, S_{L_i}. If h is analytic about L and* $\log(h)$ *is defined on S, then h does not vanish at any point of L.*

[9] Convergence in the usual Hausdorff topology for the compact subsets of C^n.

(III.3) REMARK. *It follows, that if S is simply-coconnected and rationally convex, then $S=L$.* Compare this with Question II.7.

However, there is an

(III.4) EXAMPLE. *There is a polynomially convex set, L, in C^2, such that the coordinate function, z_1, vanishes at a point of L, even though* $\log(z_1)$ *is defined on S_L.*

Such an example (in some C^n) was first constructed by K. Hoffman to refute a conjecture[10] of the author. Later, E. Bishop suggested the following, extremely simple, example. Let $E=E_1\cup E_2$, where $E_1=\{(e^{i\theta}, z_2): 0\leqq\theta\leqq\pi, |z_2|=1\}$ and $E_2=\{(e^{i\theta}, 0): \pi\leqq\theta\leqq 2\pi\}$. If we set $L=\text{hull}(E)$, then S_L is contained in E. Since E_1 and E_2 are disjoint, and $\log(z_1)$ is defined on each, it follows that $\log(z_1)$ is defined on S_L. But it is easily verified that hull(E) contains (0, 0), at which point z_1 vanishes.

Clearly, III.2 and III.4 together give a "no" answer to Question III.0.

We can also use the E of Example III.4 to show

(III.5) *There is a rational polyhedron (a compact set defined by a finite number of rational inequalities) whose polynomial convex hull is not an analytic polyhedron.*

For, E is rationally convex and is, therefore, a decreasing intersection of rational polyhedra. Hence, there is a rational polyhedron, R, which contains E and on which $\log(z_1)$ is defined. Since hull(R) must contain (0, 0), we may now apply Theorem III.1 to deduce that hull(R) is not an analytic polyhedron.

IV. **A hull with no analytic structure.** The set hull(X) is defined by a certain maximum modulus relation (see §0). Is this anything more than the classical maximum modulus principle for analytic functions on an analytic variety? In particular, *does the set* hull(X)$-X$ *consist of (or, at least, contain) positive dimensional analytic varieties?* For some fairly general cases, the analytic varieties making up hull(X)$-X$ have been exhibited. (See [**3**; **4**; **10**; **12**].) However, our result is

(IV.1) THEOREM.[11] *There is a compact set, X, in C^2, such that* $X\neq\text{hull}(X)$, *but* hull(X) *does not contain any positive dimensional analytic varieties.*

Our approach is to construct an $X\neq\text{hull}(X)$, such that neither one of the coordinate projections, $z_1(\text{hull}(X))$, $z_2(\text{hull}(X))$, contains any open subset of the plane. It follows from the open mapping property

[10] Namely, that there was no such example!

[11] A proof is given in [**8**].

of analytic functions on an analytic variety, that hull(X) cannot contain any analytic varieties (of positive dimension).

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All proofs will appear elsewhere.

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