ON EXTENDING CROSS SECTIONS IN ORIENTABLE $V_{k+m,m}$ BUNDLES

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1. **Introduction.** 1.1. Let $\mathfrak{b}=(F, p, B)$ be an orientable n-plane bundle, i.e., $p(\mathfrak{b})\colon F(\mathfrak{b})\to B(\mathfrak{b})$ is a bundle map with R^n as the fiber, $SO(n)$ as the group and a connected C.W. complex $B(\mathfrak{b})$ as base space. Let $w_i(\mathfrak{b})\in H^i(B(\mathfrak{b}); J)$ be the Stiefel-Whitney classes of $\mathfrak{b}$. The group $J=Z$ if i is odd or $i=n$ and J equals Z_2 otherwise; $w_n(\mathfrak{b})$ is also the Euler class of $\mathfrak{b}$.

1.2. Associated with $\mathfrak{b}$ are $V_{n,m}$ bundles, $\mathfrak{b}^m$ ($V_{n,m}$ is the Stiefel manifold of orthogonal m frames in R^n). Letting $k=n-m$ we will write

$$\mathfrak{b}^m = (F_{k,m}(\mathfrak{b}), B(\mathfrak{b}), p(\mathfrak{b})).$$

We are interested in finding invariants which would tell whether or not there exists a cross section in $\mathfrak{b}^m$. In a sense this problem is already solved using the Postnikov systems. What we will do is to identify certain classes in universal examples whose image in $H^*(B(\mathfrak{b}))$ must be zero if a cross section is to exist over the $k+6$ skeleton of $B(\mathfrak{b})$. The computations are based on a modification and extension of the results of Hermann [2].

The first obstruction is just $w_{k+1}(\mathfrak{b})$. It turns out that the higher classes are not unique elements but rather cosets of certain groups. We can specify this group for each obstruction studied in terms of computable operations in $H^*(B)$. In addition, these higher classes satisfy certain relations. By using both the indeterminacy and these relations, the obstructions can be computed in many interesting cases. Our detailed computation for the first six obstructions are valid only in the stable range for the homotopy groups of $V_{k+m,m}$. Full details will appear elsewhere.

Some of the applications of these results to the question of immersing and embedding manifolds into Euclidean space are listed in §5.

2. **Obstruction theory.** 2.1. Let $\mathfrak{g}=(E, p, G_{k+m})$ be the universal $k+m$ plane bundle over G_{k+m}, the Grassman manifold of oriented $k+m$ planes in R_∞. Let $\mathfrak{g}^m=(E_{k,m}, p, G_{k+m})$ be the associated $V_{k+m,m}$ bundle. Let $\mathfrak{b}=(F, p, B)$ be any $k+m$ plane bundle. To each bundle

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$\mathfrak{b}$ there is a classifying map $f_{\mathfrak{b}}: B \to G_{k+m}$ and the bundle $\mathfrak{b}^m$ has a cross section iff $f_{\mathfrak{b}}$ can be lifted to a map $f: B \to E_{k,m}$ (that is, $pf = f_{\mathfrak{b}}$). So the obstructions to finding a cross section are just the obstructions to lifting the map $f_{\mathfrak{b}}$. We will use a modification of the Postnikov tower for $\mathfrak{g}^m$ as constructed by Moore [8] and used by Hermann [2] to investigate these obstructions.

2.2. By a Postnikov tower for $\mathfrak{g}^m$ we mean a tower of fiber spaces and maps

2.2.1.
$$\begin{tikzcd}
& K(A_i, n(i)) \arrow[dl] & & & & \\
{} \arrow[r] & \overset{i}{E}_{k,m} \arrow[r, "p_i"] & \overset{i-1}{E}_{k,m} \arrow[r] & \cdots \arrow[r] & \overset{1}{E}_{k,m} \arrow[r, "p_1"] & G_{k+m} \\
& E_{k,m} \arrow[u, "h_i"] \arrow[ur, "h_{i-1}"'] \arrow[rrrr] & & & & G_{k+m}.
\end{tikzcd}$$

In the Postnikov tower of Moore, $\pi_{n(i)}(V_{k+m,m}) = A_i$, $n(i) < n(i+1)$, and each nonzero homotopy group of the fiber appears. Our modification consists in replacing the single fiber space

$$K(A_i, n(i)) \to \overset{i}{E}_{k,m} \xrightarrow{p_i} \overset{i-1}{E}_{k,m}$$

by a tower of fiber spaces if the group A_i is different from a free abelian group or a direct sum $Z_p \oplus \cdots \oplus Z_p$ for some prime p. For example, from [10] we have $\pi_{11}(V_{12,3}) = Z \oplus Z_4$ and this is the third nonzero group. The modified tower will have, in place of the single fibering of Moore's system, the following

$$\begin{tikzcd}
& K(Z_2, 11) \arrow[d] & K(Z_2, 11) \arrow[d] & K(Z, 11) \arrow[d] & \\
\cdots \arrow[r] & \overset{5}{E}_{9,3} \arrow[r] & \overset{4}{E}_{9,3} \arrow[r] & \overset{3}{E}_{9,3} \arrow[r] & \cdots .
\end{tikzcd}$$

Each fiber is constructed so that the fiber of the composite $\rho: \overset{5}{E}_{9,3} \to \overset{2}{E}_{9,3}$ will be homeomorphic to $K(Z \oplus Z_4, 11)$. This modification is important because of the difficulties involved in identifying the k-invariants if A_i is not free abelian or a vector space over a field [3].

3. **The construction of the tower 2.2.1.** In this section we will describe a procedure for constructing the modified Postnikov system.

3.1. The group A_1 is Z if $m=1$ or $k\equiv 0 \bmod 2$ and is Z_2 otherwise. The k-invariant is just w_{k+1}.

The map $h_i^*\colon H^*(E^i_{k,m})\to H^*(E_{k,m})$ is an epimorphism in dimensions up to $2k$. This implies that none of the fiber spaces in the tower (up to $n(i)=2k$) is trivial.

3.2. Suppose we have constructed $E^{i-1}_{k,m}$ such that, letting F^{i-1} be the fiber of $\rho\colon E^{i-1}_{k,m}\to G_{k+m}$, we can find a j so that $\pi_n(F^{i-1})\simeq\pi_n(V_{k+m,m})$ if $n<j$ and $\pi_n(F^{i-1})=0$ if $n\geqq j$. Then for this j and i we have

LEMMA 3.2.1. *If $n\leqq j$, then $h^*_{i-1}\colon H^n(E^{i-1}_{k,m})\to H^n(E_{k,m})$ is an isomorphism.*

COROLLARY 3.2.2. *The group*

$$B^i = \ker\left(h^*_{i-1}\colon H^{j+1}(E^{i-1}_{k,m};Z)\to H^{j+1}(E_{k,m};Z)\right)$$

is a free abelian group.

3.2.3. Let $B^i_p=\ker(h^*_{i-1}\colon H^{j+1}(E^{i-1}_{k,m};\ Z_p)\to H^{j+1}(E_{k,m};\ Z_p))$. Then 3.2.2 implies that in both B^i and B^i_p we can talk about a finite linearly independent set of generators (consider B^i_p as a vector space over Z_p). Then the space $E^i_{k,m}$ will be defined as the fiber space over $E^{i-1}_{k,m}$ having an independent set of generators of B^i as the k-invariants. The fiber will be $K(B^i, j)$. The space $E^{i+\beta}_{k,m}$ is a fiber space over $E^{i+\beta-1}_{k,m}$ having an independent set of generators of $B_2^{i+\beta}$ as k-invariants and $K(B_2^{i+\beta}, j)$ as fiber, $\beta=1, 2, \cdots$. When $B_2^{i+\beta}=0$ we continue with $B_3^{i+\beta+\beta_1}$, then with $B_5^{i+\beta+\beta_1+\beta_2}$, etc.

3.3.1. Since each class which was used as a k-invariant in the above construction has to be killed in any Postnikov tower for the fiber space $\mathfrak{g}^m$ and since no homotopy group is attached with a zero k-invariant in such a tower, the procedure of 3.1 and 3.2 will yield a modified Postnikov tower for $\mathfrak{g}^m$ up to dimension $2k$.

3.3.2. Knowledge of the homotopy groups as computed by Paechter [10] is not needed for this procedure but knowing them helps to make the computation by indicating how many elements there should be in B^i_p at each stage.

3.3.3. In order to identify the groups B^i_p the following construction is used. We first observe that $E_{k,m}$, the total space of $\mathfrak{g}^m$, is of the same homotopy type as G_k and, further, that for each i there exists a homotopy equivalence λ so that $h_i\lambda=\lambda_i$ in the following diagram:

$$\begin{tikzcd}
 & & & E_{k,m} \arrow[dl, "h_{i+1}"] \arrow[ddl, "h_i"] \\
G_k \arrow[urrr, "\lambda"] \arrow[r, "\rho"] \arrow[dr, leftrightarrow] & \tilde{G} \arrow[r, "\tilde{\lambda}"] \arrow[d] & E^{i+1}_{k,m} \arrow[d, "p_{i+1}"] & \\
 & G_k \arrow[r, "\lambda_i"] & E^{i}_{k,m}. &
\end{tikzcd}$$

In this diagram $\tilde{G}$ is the fiber space over G_k induced by λ_i from $p_{i+1}: E^{i+1}_{k,m} \to E^{i}_{k,m}$. The space $\tilde{G}$ is homeomorphic to $G_k \times K(A_{i+1}, n(i+1))$ and ρ is a cross section defined so that $h_{i+1}\lambda = \tilde{\lambda}\rho$. Then the kernel of h^*_{i+1} is just the kernel of $\rho^*\tilde{\lambda}^*$ and this group is comparatively simple to evaluate.

4. **Obstructions in sphere bundles.** 4.1. As an illustration of our procedure we will outline the result for sphere bundles. This procedure is different from the one used in [**5**] in which we followed Hermann's program more closely. We will use the symbol k^i to stand for the k-invariant of p_i: $E^{i}_{k,m} \to E^{i-1}_{k,m}$. For $m=1$ and $k \geqq 5$ (the stable range) the tower 2.2.1 begins

$$\begin{tikzcd}[column sep=small]
 & & K(Z_3, k+3) \arrow[dl] & K(Z_2, k+3) \arrow[dl] & K(Z_2, k+3) \arrow[dl] & K(Z_2, k+3) \arrow[dl] & K(Z_2, k+2) \arrow[dl] & K(Z_2, k+1) \arrow[dl] & K(Z, k) \arrow[dl] \\
{} \arrow[r] & E^7 \arrow[r] & E^6 \arrow[r] & E^5 \arrow[r] & E^4 \arrow[r] & E^3 \arrow[r] & E^2 \arrow[r] & E^1 \arrow[r] & G_{k+1} \\
 & & & & B \arrow[ul, "f_5"] \arrow[u, "f_4"] \arrow[ur, "f_3"] \arrow[urrrr, "f_{\mathfrak{b}}"'] & & & &
\end{tikzcd}$$

where $\mathfrak{b} = (F, p, B)$ is a k-sphere bundle. The first obstruction, k^1, is just $w_{k+1}(\mathfrak{b})$. If $w_{k+1}(\mathfrak{b}) = f^*_{\mathfrak{b}} w_{k+1} = 0$ then f_1 can be defined. In general, f_i is defined iff there exists an f_{i-1} such that $f^*_{i-1} k^i = 0$.

THEOREM 4.1.1 (LIAO [4]). *As f_1 ranges over all possible liftings, $f_1^* k^2$ ranges over a coset of $(Sq^2 + w_2(\mathfrak{b}) \cdot) H^k(B; Z) \subset H^{k+2}(B; Z_2)$. We also have $(Sq^2 + w_2(\mathfrak{b}) \cdot) k^2 = 0$.*

THEOREM 4.1.2. *For a fixed f_1, as f_2 ranges over all possible liftings, $f_2^* k^3$ ranges over a coset of $(Sq^2 + w_2(\mathfrak{b}) \cdot) H^{k+1}(B; Z_2) = G_1 \subset H^{k+3}(B; Z_2)$.*

THEOREM 4.1.3. *There is a class* $k^{4'} \in H^{k+4}(E^1, Z_2)$ *such that* $p_3^* p_2^* k^{4'} = k^4$. *We have* $Sq^1 k^{4'} + (Sq^2 Sq^1 + w_3(\mathfrak{b})\cdot)k^2 = 0$ *in* $H^*(E^1)$.

THEOREM 4.1.4. *The class* k^5 *satisfies* $Sq^1 k^5 = 0$.

THEOREM 4.1.5. *As* f_3, f_4 *and* f_5 *range over all possible liftings such that* $f_3^*(k^4) = f_4^*(k^5) = 0$, *the class* $f_5^* k^6$ *lies in a coset of a group* K *containing* $(Sq^2 + w_2(\mathfrak{b})\cdot)H^{k+2}(B; Z_2) \oplus Sq^1 H^{k+3}(B; Z_2) = K_1$.

THEOREM 4.1.6. *There is a class* $k^{7'} \in H^{k+4}(E^1, Z_3)$ *such that its image in* $H^{k+4}(E^6; Z_3)$ *is* k^7. *As* f_1 *ranges over all possible liftings* $f_1^*(k^{7'})$ *lies in a coset of* $(\mathcal{P}_3^1 + \mathfrak{p}_1)H^k(B; Z) \subset H^{k+4}(B; Z_3)$. ($\mathcal{P}_3^1$ *is the first reduced Steenrod cube and* $\mathfrak{p}_1$ *is the first Pontryagin class* (mod 3) *of* $\mathfrak{b}$.)

4.2. If we vary f_1 and f_2 such that $f_1^* k^2 = 0$, then $f_2^* k^3$ lies in a coset of a group G containing G_1. This group is identified in [5] in terms of a "secondary bundle operation." If we vary f_i, $i = 1, \cdots, 5$ such that $f_i^* k^{i+1} = 0$, $i = 1, \cdots, 4$ then $f_5^* k^6$ lies in a coset of a group K containing K_1. This group is also identified in [5] in terms of a "tertiary bundle operation." For all the applications we have made, it has been sufficient to know the relations and indeterminacy as described in 4.1.1 to 4.1.6.

4.3. As an illustration on how one can apply these results we will prove

THEOREM 4.3.1. *Suppose* $k \equiv 7$ mod 8. *If* $\mathfrak{b}$ *is any orientable* $k+1$ *plane bundle over* P_{k+6} *(real projective space), such that* $w_2(\mathfrak{b}) \neq 0$ *and* $w_{k+1}(\mathfrak{b}) = 0$, *then* $\mathfrak{b}$ *has a nonzero cross section.*

PROOF. Since $H^k(P_{k+6}; Z) = 0$, the indeterminacy of $f_1^* k^2$ is zero. Since $(Sq^2 + w_2\cdot)H^{k+2}(P_{k+6}; Z_2) \to H^{k+4}(P_{k+6}; Z_2)$ is an isomorphism, the second part of 4.1.1 implies $f_1^*(k^2) = 0$. Since Sq^1 is an isomorphism in dimension $k+4$, 4.1.3 implies $f_1^*(k^{4'}) = 0$. For fixed f_1, $f_2^* k^3$ is in a coset of $(Sq^2 + w_2)H^{k+1}(P_{k+5}; Z_2) = H^{k+3}(P_{k+5}; Z_2)$ and hence there is a lifting f_2 such that $f_2^*(k^3) = 0$. Since $f_1^*(k^{4'}) = 0$, we have $f_3^*(k^4) = 0$ for any choice of f_3. By 4.1.4 we see that for any choice of f_4, $f_4^*(k^5) = 0$. Since $(Sq^2 + w_2\cdot)H^{k+2}(P_{k+6}; Z_2) = H^{k+4}(P_{k+6}; Z_2)$, by 4.1.5 there is a choice of f_3, f_4 and f_5 such that $f_5^* k^6 = 0$. Finally, since $H^{k+4}(P_{k+6}; Z_3) = 0$, $f_6^* k^7 = 0$. Since $\pi_{k+4}(S^k) = \pi_{k+5}(S^k) = 0$ the obstructions in the next two dimensions are necessarily zero.

5. Applications.

LEMMA 5.1. *Let* $\mathfrak{b}$ *be any orientable* $k+1$ *plane bundle over* B *and* $\mathfrak{b}_m = \mathfrak{b} \oplus mI$ *(*mI *is the trivial* m *plane bundle over* B*). If* $k \equiv 0$ mod 4,

$\mathfrak{b}^1$ has a cross section over[2] *$B_{(k+3)}$ iff $\mathfrak{b}_m^{m+1}$ has a cross section over $B_{(k+3)}$ for some m. If $k\equiv 2 \bmod 4$ $\mathfrak{b}^2$ has a cross section over $B_{(k)}$ iff $\mathfrak{b}_m^{m+2}$ has a cross section over $B_{(k)}$ for some m.*

There are other results of this type but this Lemma, using Proposition 2 of [1], yields

THEOREM 5.2. *The following embeddings of real projective spaces are possible*: (a) *If $k=3$ mod 4, $k>3$, then $P_k\subset R^{2k-2}$.* (b) *If $k=4q+i$, $i=0$, 1, or 2, and $q\neq 2^j$ or 0, then $P_k\subset R^{2k-3}$.*

This result answers in the negative the conjecture in [1], attributed to Atiyah, on the minimum dimension Euclidean space into which a projective space embeds.

Let $\mathfrak{b}=(E, B, p)$ be an orientable $k+m$ plane bundle. Let $U(\mathfrak{b})$ be the Thom class of $\mathfrak{b}$, i.e., $U(\mathfrak{b})\in H^{k+m}(E, E_0; Z)$ (E_0 is the collection of nonzero vectors). With these definitions we have

THEOREM 5.3. *For each secondary obstruction coset $v(k, \mathfrak{b})$ to finding a cross section of $\mathfrak{b}^m$, $m\geqq 3$, there is a secondary cohomology operation ϕ such that $\phi(U)=U\cdot(v(k, \mathfrak{b})+\alpha)$ where α is polynomial in the $w_i(\mathfrak{b})$.*

We have been able to show that α is either $w_2\cdot w_k$ or zero and, if $k=2^j-2$, then $\alpha=w_2\cdot w_k$. Under greater restrictions, a similar formula holds for some of the other higher obstructions.

Theorem 5.3 can be applied to the question of immersing differential manifolds. Hirsch [3] has proved that, if $\mathfrak{n}$ is the normal bundle to an embedding of an orientable compact differentiable manifold M^n without boundary into Euclidean space R^m and if $\mathfrak{n}^r$ has a cross section, then M^n can be immersed in R^{m-r}. If we apply 5.3 to the normal bundle of an embedding of M^n into R^{2n+1}, using some of the ideas of Massey in [6], we obtain the first of the following theorems. The second follows by a refinement of the argument. In the next three theorems $\mathfrak{n}$ is this normal bundle.

THEOREM 5.4. *Suppose $n>4$. If $n\equiv 0$ mod 2, then M^n is immersible in R^{2n-2} iff $w_2(\mathfrak{n})\cdot w_{n-2}(\mathfrak{n})=0$. If $n\equiv 0$ mod 4 and $w_{n-2}(\mathfrak{n})=0$, then M^n is immersible in R^{2n-3}.*

This theorem contradicts a result of Novikov (Theorem 2 of [9]). Indeed if M^n is complex projective space $CP_{n/2}$ where $n/2=2^j$ then $w_2(\mathfrak{n})\cdot w_{n-2}(\mathfrak{n})\neq 0$ and so $M^n\not\subseteq R^{2n-2}$ (M^n does not immerse in R^{2n-2}). But Novikov's result implies $M^n\subseteq R^{2n-2}$. I understand that J. Levine has an alternate proof of this result. In addition Massey [7] has shown that if $w_{n-2}(\mathfrak{n})\neq 0$ then $n=2^k(2^h+1)$ for non-negative integers

[2] The notation $B_{(n)}$ denotes the n-skeleton of B.

h and k, with cases $k=1$, $h>0$ and $h=1$, $k\geqq 0$ ruled out. Thus $w_2(\mathfrak{n})\cdot w_{n-2}(\mathfrak{n})=0$ except possibly for these values of n.

THEOREM 5.5. *Let* $A=Sq^1H^{n-2}(M^n;\ Z_2)$, $n>5$ *and* $w_{n-2}(\mathfrak{n})=0$. *If* $n\equiv 1 \bmod 4$, *then the second obstruction to finding a cross section in* $\mathfrak{n}^4$ *is* $\alpha+A$. *If* $n\equiv 3 \bmod 4$, *then, the second obstruction differs from* α *by a class in* A. *If* $n\equiv 0 \bmod 4$, *the second obstruction is zero.*

If $n\equiv 1 \bmod 4$, the third obstruction to finding a cross section to $\mathfrak{n}^4$ is always zero by [10]. Therefore we have

THEOREM 5.6. *If* $n=1+2^i$, $n>5$, $w_{n-2}(\mathfrak{n})=0$, *then* M^n *is immersible in* R^{2n-3} *iff* $w_2(\mathfrak{n})\cdot w_{n-3}(\mathfrak{n})\in Sq^1H^{n-2}(M^n;\ Z_2)$.

Finally, applying Hirsch's theorem to projective spaces, we have

THEOREM 5.7. *The following results on the immersion of projective spaces hold*:

(a) *If* $n\equiv 1 \bmod 4$, $n\neq 2^i+1$, *then* $P_n\subseteq R^{2n-4}$.
(b) *If* $n\equiv 2^i+1$, $n>3$, *then* $P_n\subseteq R^{2n-3}$.
(c) *If* $n\equiv 3 \bmod 8$, $n\geqq 19$, *then* $P_n\subseteq R^{2n-6}$.
(d) $P_{15}\nsubseteq R^{20}$.

By quite different methods, B. J. Sanderson and J. Levine have obtained (a), (b), and (c). Only is (b) known to be the best possible result.

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