DISCRETE TRANSFORMATIONS ON TORI AND FLOWS ON SOLVMANIFOLDS

BY L. AUSLANDER[1] AND F. HAHN[2]

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In an Annals Study [1] (which is to appear shortly) the authors and L. Green, with several other contributors, studied the ergodic behavior of one parameter groups acting on special classes of homogeneous spaces. Essentially these homogeneous spaces were always solvmanifolds with discrete isotropy groups. In this note we will announce some results on solvmanifolds where the isotropy group need not be discrete but merely closed. However, the class of solvmanifolds which we can treat in such generality at this time is quite restricted. To make this restriction precise, they are the class of compact solvmanifolds whose fundamental groups have the form $Z \cdot Z^s$, where Z denotes the integers and the dot denotes the semi-direct product. By a known structure theorem of Mostow [3] all such manifolds are fiber bundles with fiber the torus, base space the circle, and the group of the bundle is the group generated by an element of the affine group of the torus. By the affine group of the torus we mean the group of all homeomorphisms of the torus which may be written as an automorphism followed by a translation. This group is isomorphic to $\mathfrak{A}(T) \cdot T$ where $\mathfrak{A}(T)$ is the automorphism group of the torus.

This situation enables us to reduce the classification of these flows to the classification of a single affine transformation acting on the torus. This problem is a generalization of a classic result of Halmos [2] and our solution involves only slight modification of his methods.

After this classification of the affine transformations of the torus we examine the following situation. Let $\Gamma = Z \cdot Z^s$ and let S be a solvable Lie group with a closed subgroup C such that S/C is compact and $\Pi_1(S/C) = \Gamma$. (S is not uniquely determined by these conditions.) Let z be a generator of Z and let $\psi \in \mathfrak{A}(T) \cdot T$ be the generator of the group of the bundle, S/C over the circle, to which z corresponds. The affine transformation ψ factors $\psi = \tau \circ \alpha$, where α is an automorphism of T and τ a translation of T. If we let $T = R^n/L$ where L is the integral lattice of R^n then we see that to each automorphism α of T there corresponds a linear transformation α of R^n which leaves L invariant. We call this linear transformation α an automorphism associated

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with S/C. We let α^* denote the adjoint of α and for each integer $p \geqq 1$ we let

$$K_p = [\text{Kernel } ((\alpha^*)^p - I)] \cap L.$$

We note that $K_1 \subset K_p$ for each $p \geqq 1$. In this language our main result may be stated as follows.

THEOREM 1. *Let* $\Gamma = Z \cdot Z^s$ *and let* S *be a solvable Lie group with a closed subgroup* C *such that* S/C *is compact and* $\Pi_1(S/C) = \Gamma$. *If* α *is an automorphism associated with* S/C *then the following statements hold*:

1. *If* $\{0\} = K_p$ *for all* $p \geqq 1$ *then there exists a one parameter group which does not lie in the maximal nilpotent normal subgroup of* S *which induces an ergodic flow on* S/C.

2. *If* $\{0\} \neq K_1 = K_p$ *for each* $p \geqq 1$ *then there exist one parameter subgroups of* S *which induce ergodic flows on* S/C *if and only if there is a characteristic vector, characteristic value* 1 *for all of* ad(S), *which lies in general position in* K_1 *relative to* $K_1 \cap L$.

3. *If there is a* $p > 1$ *such that* $K_1 \neq K_p$ *then no one parameter subgroup of* S *induces an ergodic flow on* S/C.

We observe that if α has no eigenvalues which are roots of unity then we have case 1.

Complete proofs will be presented elsewhere.

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YALE UNIVERSITY AND
INDIANA UNIVERSITY