ANALYTIC MAPPINGS BETWEEN ARBITRARY RIEMANN SURFACES[1]

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1. **The problem.** L. Ahlfors [1] established the first and second main theorems for analytic mappings from the plane R into a closed Riemann surface S. Recently S. Chern [3] extended these theorems to the case where R is a closed Riemann surface less a finite number of points, and S is closed. He also announced a generalization to an R that is obtained from a closed surface by removing a finite number of points and disks. Here we shall consider an arbitrary open surface R, of finite or infinite genus, and an arbitrary closed or open S.

Chern's elegant method made substantial use of results in related fields: differential geometry, projective geometry, topology, and partial differential equations. Our approach is elementary and purely analytic: the only tools used are the principal functions.

The theorems will be formulated so as to include earlier results. In the classical case of the plane or the disk we obtain a new form of the second main theorem where exceptional intervals are not needed.

2. **Singularity function.** Let $a_1, \cdots, a_q, \zeta_0, \zeta_1$, be points on an arbitrary closed or open Riemann surface S with local variable ζ. Construct the principal function t_0 with singularities $-c \log|\zeta-\zeta_0|$ and $c \log|\zeta-\zeta_1|$, $0<c$ (cf. [2; 4]). Near the ideal boundary choose $t_0=L_0t_0$ where L_0 is the operator associated with vanishing normal derivative. Normalize t_0 by $t_0(\zeta)+c \log|\zeta-\zeta_0|\to 0$ as $\zeta\to\zeta_0$, and set $s_0=\log(1+e^{t_0})$.

For $a\neq\zeta_0$ let $t=t(\zeta, a)$ be the principal function with a harmonic $t(\zeta, a)+c \log|\zeta-a|$ at a. At ζ_0 let $t(\zeta, a)-c \log|\zeta-\zeta_0|$ be harmonic and tend to $s_0(a)$ as $\zeta\to\zeta_0$. Again choose the boundary behavior $t=L_0t$. The function $s(\zeta, a)=s_0(\zeta)+t(\zeta, a)$ is symmetric:

$$s(a, b) = s(b, a). \tag{1}$$

Endow S with the conformal metric $\lambda(\zeta)|d\zeta|$ where

$$\lambda^2 = \Delta s = \frac{e^{t_0}|\operatorname{grad} t_0|^2}{(1+e^{t_0})^2}. \tag{2}$$

The total area μ of S is $\leqq 2\pi c$ and the Gaussian curvature is $K=1$.

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Let $d\omega(\zeta)$ be the Euclidian area element of S and set $d\mu(\zeta)=\lambda^2(\zeta)d\omega(\zeta)$.

3. **First main theorem.** Consider an arbitrary open Riemann surface R and an analytic mapping $\zeta=f(z)$ of R into S. Let R_0 be a compact bordered subregion [2] of R with border β_0 such that $f(z)$, $z\in\beta_0$, is different from $a_1, \cdots, a_q, \zeta_0, \zeta_1$, and from the zeros of λ. Let Ω be a relatively compact bordered region in $R-\overline{R}_0$ with border $\beta_0+\beta_\Omega$.

Consider the subregion Ω_h of Ω, bordered by β_0 and the level line β_h: $v=h$, $0\leqq h\leqq k$. Form the harmonic function u on $\overline{\Omega}_h$ with $u=0$ on β_h, $u=h=\text{const.}>0$ on β_0, $\int_{\beta_0}du^*=-1$, and set $v=h-u$. For any function ϕ in Ω denote by $n(h,\phi)$ the number of zeros of ϕ in Ω_h. We set $d\mu_z=|f'(z)|^2\lambda(f(z))dw_z$, where dw_z is the Euclidian area element in R, designate symbolically by $n(h, f-a)$ the number of a-points of f in Ω_h, and introduce:

$$
\begin{aligned}
A(h, a) &= 2\pi c\int_0^h n(h, f-a)dh,\\
B(h, a) &= \int_{\beta_h} s(f(z))dv^*;\\
C(h) &= \int_{\Omega_h} u(z)d\mu_z,\\
D(h, a) &= B(0, a)+hB'(0, a).
\end{aligned}
\tag{3}
$$

Here $A(h, a)$ serves as the counting function, $B(h, a)$ as the proximity function, and $C(h, a)$ as the characteristic.

Green's formula applied to u and s gives:

FIRST MAIN THEOREM. *For arbitrary analytic mappings and every* $\Omega\subset R$,

$$A(k, a)+B(k, a)=C(k)+D(k, a). \tag{4}$$

The only functions of interest will be those with $k/C(k)\to 0$ as $\Omega\to R$, and for such functions $D(k, a)$ is a negligible remainder.

Stokes formula applied to ds^* gives

$$2\pi cn(h, f-a)+\int_{\beta_h-\beta_0}ds^*=\int_{\Omega_h}d\mu_z,$$

showing that $C'(h)=\int_{\Omega_h}d\mu_z$.

4. **Second main theorem.** Had we normalized μ (instead of K) to be unity, an integration of (4) with respect to such $d\mu_0$ would give

$\int_S B(h, a)d\mu_0 = O(h)$. Thus the points a that give substantial contributions to the proximity function are exceptional. This suggests a second main theorem even in the present general situation.

For any function $\phi(h, a)$ let $\phi(h) = \sum_1^q \phi(h, a_i)$. Then

$$A(h) + B(h) = qC(h) + D(h). \tag{5}$$

Let $s_i(\zeta) = s(\zeta, a_i)$, $\sigma = \exp\left(\sum_1^q s_i/(1+\epsilon)\right)$, $\epsilon > 0$, and

$$\begin{aligned} G(h) &= \int_{\beta_h} \log(\sigma\delta^2)dv^*, \\ H(h) &= -2\int_{\beta_h} \log \delta dv^*, \end{aligned} \tag{6}$$

where δ is the density induced in the $(v+iv^*)$-plane by λ,

$$\delta(z) = \lambda(f(z))\,|f'(z)|\,|\operatorname{grad} v|^{-1}. \tag{7}$$

Then $B(h) = (1+\epsilon)(G(h)+H(h))$.

We obtain, by first evaluating $H'(h)$ and then integrating:

$$\begin{aligned} H(h) = H(0) &+ 4\pi[-N(h,\lambda) - N(h,f') + E(h)] \\ &+ (c-2)\pi[N(h,\zeta_0) + N(h,\zeta_1)] + 2C(h). \end{aligned} \tag{8}$$

Here $N = \int_0^h n dh$, e is the Euler characteristic of Ω_h, and $E = \int_0^h e dh$.

For any function ϕ in $(0, k)$ set $\phi_2(h) = \int_0^h\int_0^y \phi(x)dxdy$. In terms of $I(h) = \int_{\beta_h} \sigma\delta^2 dv^*$ we have by the convexity of the logarithm

$$G_2(h) < h^2 \log I_2(h) + O(h^3). \tag{9}$$

The distribution $dm = \sigma d\mu_z$ gives finite mass $m = \int_S dm$ and the integral $M(\zeta) = \int_S s(\zeta, a)dm(a)$ converges by (1). Integration of (4) with respect to $dm(a)$ over S gives

$$2\pi c I_2(h) = mC(h) - M(h) + O(h) \tag{10}$$

where $M(h) = \int_{\beta_h} M(\zeta)dv^*$.

On substituting into (5), redenoting 2ϵ by ϵ, and letting $c=2$, $h=k$, one obtains the following implicit form of the

SECOND MAIN THEOREM. *For every $\Omega \subset R$, and arbitrary R and S,*

$$\begin{aligned} (q-2-\epsilon)C(k) < 4\pi[N(k) &- N(k,\lambda) - N(k,f') + E(k)] \\ &+ G(k) + O(k), \end{aligned} \tag{11}$$

where $N(k) = \sum_1^q N(k, f-a_i)$ and $G(k)$ satisfies (9), (10).

5. **Applications.** Add to $a_1 \cdots a_q$ points $a_{q+1}, \cdots, a_{q+p}$ among the

zeros of λ including at least all those covered by $f(\Omega)$. Then $\sum_{q+1}^{q+p} N(k, a_i) - N(k, \lambda) = 0$ and

$$(12)\quad \begin{aligned}(q + p - 2 - \epsilon)C(k) < 4\pi\left[\sum_{1}^{q} N(k, a_i) - N(k, f') + E(k)\right] \\ + G(k) + O(k).\end{aligned}$$

Instead of giving an explicit estimate, with exceptional intervals, for G from G_2 it is simpler to replace B by the integral B_2 of the "areal" proximity $\int_0^h B dh$. The scarcity of points a contributing to B_2 remains valid when A, C, D in (5) are replaced by A_2, C_2, D_2. In (12) $O(k)$ is replaced by $O(k^3)$, the subindex "2" is attached to all other capital letters, and $G_2(k)$ is directly given by (9), (10).

If S is closed, then $M(\zeta)$ is bounded on S and $M(k) = O(1)$ in $k = k(\Omega)$, $\Omega \to R$. There are $p = 2g$ zeros of λ, and $p - 2$ is the Euler characteristic e_S of S. We have the following

COROLLARY. *In an analytic mapping of an arbitrary open R into a closed S,*

$$(13)\quad (q + e_S - \epsilon)C_2(k) < 4\pi[N_2(k) - N_2(k, f') + E_2(k)] + G_2(k),$$

where

$$(14)\quad G_2(k) = O(k^2 \log C(k) + k^3).$$

We choose the defect

$$(15)\quad \delta(a) = 1 - \limsup_{\Omega \to R} \frac{4\pi N_2(k)}{C_2(k)}$$

and set

$$(16)\quad \eta = \limsup_{\Omega \to R} \frac{4\pi E_2(k)}{C_2(k)} \cdot$$

For functions with $G_2(k)/C_2(k) \to 0$ we obtain the following.

DEFECT RELATION.

$$(17)\quad \sum \delta(a_i) \leqq \eta - e_S.$$

In particular, there can be at most $\eta - e_S$ Picard values.

Functions for which C_2 grows more rapidly than E_2 cannot give a mapping into an S with $g > 1$. For an R that is a closed surface punctured at a finite number of points this is Chern's theorem.

In the special case where R has a capacity function p_β with compact level lines, this can be taken for v, and the directed limits for $\Omega \to R$

replaced by limits as $k \to \sup_R p_\beta$. By l'Hopital's rule the quantities C_2, N_2, E_2 in (15), (16) can then be replaced by C, N, E, and we find the usual forms.

For meromorphic functions in the plane or the disk take $v = (1/2\pi) \log r$. Then (13) is a new form of the second main theorem that holds without exceptional intervals.

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